Gravity and Gauge Theory

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Abstract

In this master thesis, the goal is to shed some light on the relations and differences between gravity and gauge theory. The Einstein-Cartan formulation of general relativity is reviewed and the three-dimensional theory is formulated as a Chern-Simons gauge theory. The four-dimensional case is briefly discussed.

Contents

1 Introduction

What is the connection between gravity and gauge theory? This was the question that led to this master thesis. While it is true that some similarities with the gauge theories are already visible in the theory of General Relativity as it was originally set up by Einstein, most of these common features are somewhat buried inside the formalism of tensor calculus.

Gravity is a theory which takes a manifold and endows it with a geometry. This geometry is usually described in terms of a (pseudo)-Riemannian metric which is the fundamental degree of freedom in Einstein gravity. In contrast, most gauge theories are set up on a background metric. Moreover, gauge theories describe the dynamics of a connection (the gauge field) while the Levi-Civita connection in Einstein gravity has no dynamics of its own.

The reformulation of gravity as a gauge theory thus happens in several steps. The different conceptual changes will be described in detail in the following chapters. Chapter 2 will deal with recasting Einstein gravity in a language more amenable to the gauge approach. First, we will replace the metric, which is a complicated object, by a simpler one, namely the metric of a flat space of the same signature. In a curved space, no coordinate transformation can accomplish this, but we can get this simplification by using vielbeins, which will be one part of the new fundamental variables. This will introduce a gauge freedom of local frame rotations preserving the flat metric. Secondly, the use of differential forms will make it possible to simplify most of the computational issues by enabling covariant differentiation without needing a metric. The final step which then paves the way towards a gauge theory of gravity will be to pass from the Levi-Civita connection to a more general one. This so-called spin connection will have new degrees of freedom living in a Lie algebra (akin to a gauge connection). The Palatini action of gravity which is used in this approach is first-order in the fundamental variables, and the torsion-free connection will be retrieved dynamically. This entire framework is called the Einstein-Cartan formulation of gravity.

The idea which leads to gauge gravity is a unification of the spin connection and the vielbeins in a gauge field of some larger group. This is the easy part. The difficult part is to find a background-independent and gauge invariant action which is equivalent to the Palatini action. In the third chapter, this will be carried out in three dimensions, where the Chern-Simons action fills this purpose. The Chern-Simons formulation of gravity is very simple because gravity in three dimensions has no local degrees of freedom. Nevertheless, there are some subtleties concerning gauge transformations. The larger group means that Chern-Simons gravity has more symmetries than we are accustomed to. In particular, every classical solution of Chern-Simons gravity is a flat connection and therefore gauge equivalent to the trivial solution. Since the trivial solution represents a degenerate geometry, it is unclear what the gauge symmetry actually means in the context of gravity.

In the last chapter, an attempt will be made at using this approach in four dimensions. The kinematical part is as easy as in three dimensions, and Cartan geometry explains this very nicely. Finding a suitable action principle however is presently an open problem. One possibility is the MacDowell-Mansouri formulation, but the action is not gauge invariant. It is however possible to set the theory up for spontaneous symmetry breaking. The ensuing Stelle-West model will also be presented.

1.1 Literature

Most of the matters discussed in this thesis can be found in various articles and books of which I would like to list the ones that were particularly helpful.

The first-order formulation of gravity is reviewed in the first two chapters of [10]. The Chern-Simons formulation in 2+1 dimensions is presented in [9].

The mathematics of Cartan geometry and its relation to the gauge theory formulations of gravity are discussed by Wise in [7]. The mathematical background can be found in [6]. A shorter discussion can be found in [8], which also covers the scalar products on the six-dimensional Lie algebras of Chern-Simons gravity. An interesting treatment of gauge transformations and diffeomorphisms is found in Matschull's paper [4].

A very good paper on gauge gravity by Randono [5] includes MacDowell-Mansouri and the Stelle-West model. Randono also touches on aspects of quantization and is therefore recommended if one wants to go beyond the scope of this thesis. For the quantization in 2+1 dimensions, the work of Carlip has to be mentioned, in particular [1, 2].

2 Einstein-Cartan Gravity

The theory of gravity is built on the principle of General Covariance under coordinate transformations. For this reason, it is usually presented in the language of tensor calculus.

In this chapter, a quick review of General Relativity is given first. Then, the theory is re-expressed in a more coordinate-independent manner by introducing vielbeins and differential forms. Lastly, the new language is used to write down a first order action which yields the Einstein equations in arbitrary dimensions.

2.1 Review of General Relativity

The starting point for our discussion of general relativity is that spacetime is a pseudo-Riemannian manifold, equipped with a metric tensor or line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2.1}$$

which defines the effect of the gravitational force on freely falling observers through the action principle $\delta \int d\tau = 0$. The resulting Euler-Lagrange equations, also known as the *geodesic equations*,

$$\ddot{x}^\lambda(\tau) + \Gamma^\lambda{}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0 \tag{2.2}$$

contain the metric tensor through the Christoffel symbols

$$\Gamma^\lambda{}_{\mu\nu} = \tfrac{1}{2} g^{\lambda\rho} \left(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu} \right) \tag{2.3}$$

which are the components of the so-called Levi-Civita connection. The latter is singled out from an infinite set of connections by two conditions, namely *metric compatibility* and the *vanishing of torsion*. Let us recall what these conditions mean and how they are uniquely solved by the Levi-Civita connection.

An arbitrary affine connection $\omega^\lambda{}_{\mu\nu}$ need not be related to any metric structure. However, it defines a covariant derivative D_ν given by the following action on a vector field[1]

$$D_\nu V^\mu = \partial_\nu V^\mu + \omega^\mu{}_{\nu\rho} V^\rho \tag{2.4}$$

[1]The action on an arbitrary tensor follows from the requirement that D_ν satisfy the Leibniz rule.

This derivative is tensorial provided that under coordinate transformations $x^\mu \to y^\mu$ the connection transforms as

$$\omega^\lambda{}_{\mu\nu} \to \frac{\partial x^\sigma}{\partial y^\nu}\frac{\partial x^\rho}{\partial y^\mu}\left(\omega^\gamma{}_{\rho\sigma}\frac{\partial y^\lambda}{\partial x^\gamma} - \frac{\partial}{\partial x^\sigma}\frac{\partial y^\lambda}{\partial x^\rho}\right) \tag{2.5}$$

The geometric significance of any connection is that it provides a notion of *parallelism*. A quantity T (scalar or tensor) is said to be parallel transported along a curve $x^\mu(\tau)$ if $\dot{x}^\nu D_\nu T = 0$. And this is the point where the metric potentially comes into play. If there is a metric on the manifold, it is natural to assume that the connection is in some sense compatible with it. Commonly, one assumes that the angle between two vectors (as given by the metric tensor) should be preserved by parallel transport along any curve, i.e.

$$\dot{x}^\rho D_\rho (g_{\mu\nu} V^\mu W^\nu) = 0 \tag{2.6}$$

which is only possible if the metric is covariantly constant, i.e. $D_\rho g_{\mu\nu} = 0$. If this is spelt out, the condition of metricity reads

$$\partial_\rho g_{\mu\nu} - \omega^\sigma{}_{\rho\mu} g_{\sigma\nu} - \omega^\sigma{}_{\rho\nu} g_{\mu\sigma} = 0 \Leftrightarrow \partial_\rho g_{\mu\nu} = \omega_{\nu\rho\mu} + \omega_{\mu\rho\nu} \tag{2.7}$$

This does not yet suffice to single out a unique connection. Therefore, another requirement is usually postulated, namely that parallel transport should be path-independent for scalar quantities, that is,

$$[D_\mu, D_\nu]\,\phi \overset{!}{=} 0 \tag{2.8}$$

Since the covariant derivative reduces to the ordinary one on scalars, expanding the right-hand side leads to

$$\begin{aligned} [D_\mu, D_\nu]\,\phi &= D_\mu \partial_\nu \phi - D_\nu \partial_\mu \phi \\ &= \partial_\mu \partial_\nu \phi - \partial_\nu \partial_\mu \phi - \left(\omega^\rho{}_{\mu\nu} - \omega^\rho{}_{\nu\mu}\right) \partial_\rho \phi \\ &= -\left(\omega^\rho{}_{\mu\nu} - \omega^\rho{}_{\nu\mu}\right) \partial_\rho \phi \end{aligned} \tag{2.9}$$

where in the last step we used the fact that ordinary partial derivatives commute. It turns out that (2.8) can be satisfied only if the connection is symmetric in the two lower indices,

$$\omega^\rho{}_{\mu\nu} = \omega^\rho{}_{\nu\mu} \tag{2.10}$$

or, equivalently, if the torsion tensor vanishes,

$$T^\rho{}_{\mu\nu} = \omega^\rho{}_{\mu\nu} - \omega^\rho{}_{\nu\mu} = 0 \tag{2.11}$$

and that is why this condition is called the 'no-torsion' condition.

Together, equations (2.7) and (2.10) imply that ${\omega^\lambda}_{\mu\nu} = {\Gamma^\lambda}_{\mu\nu}$ as defined in (2.3)[2]. For the rest of this section, we work exclusively with the Levi-Civita connection and call its covariant derivative ∇_μ. As already mentioned, the second covariant derivatives do not commute on vectors. Instead, the commutator is proportional to the same vector

$$[\nabla_\mu, \nabla_\nu] V^\lambda = {R^\lambda}_{\sigma\mu\nu} V^\sigma \tag{2.14}$$

with ${R^\lambda}_{\sigma\mu\nu}$ the *Riemann Curvature Tensor*. An explicit calculation shows that

$${R^\lambda}_{\sigma\mu\nu} = \partial_\mu {\Gamma^\lambda}_{\sigma\nu} - \partial_\nu {\Gamma^\lambda}_{\sigma\mu} + {\Gamma^\lambda}_{\rho\mu}{\Gamma^\rho}_{\sigma\nu} - {\Gamma^\lambda}_{\rho\nu}{\Gamma^\rho}_{\sigma\mu} \tag{2.15}$$

The Einstein-Hilbert action

The problem of finding a suitable action which yields the Einstein equations is essentially solved by considering the simplest scalar which contains at most second derivatives of the metric. One can obtain such a quantity from contractions of the Riemann Tensor. The only nontrivial contraction is

$$R_{\mu\nu} = {R^\lambda}_{\mu\lambda\nu} \tag{2.16}$$

which is called the Ricci tensor. Taking its trace, one arrives at the Ricci scalar,

$$\mathcal{R} = g^{\mu\nu} R_{\mu\nu} = {R^{\mu\nu}}_{\mu\nu} \tag{2.17}$$

One can also add a dimensionful constant Λ, and write down the Einstein-Hilbert action with cosmological constant,

$$S_{\mathrm{EH}}[g_{\mu\nu}] = \int d^4x \sqrt{g}(\mathcal{R} - 2\Lambda) \tag{2.18}$$

It can be shown that variation of this action with respect to the metric yields the vacuum Einstein equations with cosmological constant

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}\mathcal{R} g_{\mu\nu} = -\Lambda g_{\mu\nu} \tag{2.19}$$

[2] The trick to prove this is to use the fact that the Levi-Civita connection *is* metric compatible and torsion-free,

$$\partial_\rho g_{\mu\nu} = \Gamma_{\nu\rho\mu} + \Gamma_{\mu\rho\nu}, \quad {\Gamma^\rho}_{\mu\nu} = {\Gamma^\rho}_{\nu\mu} \tag{2.12}$$

and to do some fancy manipulations

$$\begin{aligned}
\omega_{\rho\mu\nu} &\overset{(2.10)}{=} \tfrac{1}{2}\left(\omega_{\rho\mu\nu} + \omega_{\nu\mu\rho}\right) + \tfrac{1}{2}\left(\omega_{\rho\nu\mu} + \omega_{\mu\nu\rho}\right) - \tfrac{1}{2}\left(\omega_{\nu\rho\mu} + \omega_{\mu\rho\nu}\right) \\
&\overset{(2.7)}{=} \tfrac{1}{2}\partial_\mu g_{\nu\rho} + \tfrac{1}{2}\partial_\nu g_{\mu\rho} - \tfrac{1}{2}\partial_\rho g_{\mu\nu} \\
&\overset{(2.12)}{=} \tfrac{1}{2}\left(\Gamma_{\rho\mu\nu} + \Gamma_{\nu\mu\rho}\right) + \tfrac{1}{2}\left(\Gamma_{\rho\nu\mu} + \Gamma_{\mu\nu\rho}\right) - \tfrac{1}{2}\left(\Gamma_{\nu\rho\mu} + \Gamma_{\mu\rho\nu}\right) \\
&\overset{(2.12)}{=} \Gamma_{\rho\mu\nu}
\end{aligned} \tag{2.13}$$

but we shall derive this in the simpler framework of differential forms (see equations (2.87) to (2.114)). There is another way of writing the vacuum Einstein equations. By tracing the Einstein tensor, one finds

$$\mathcal{R} = \frac{2n}{n-2}\Lambda \tag{2.20}$$

and one can substitute this back into the Einstein equation, which leads to

$$R_{\mu\nu} = \frac{2}{n-2}\Lambda g_{\mu\nu} \tag{2.21}$$

2.2 Vielbeins

Instead of working with the cotangent space of the manifold, one uses an isomorphism to move to an auxiliary vector space

$$e^a = e^a{}_\mu dx^\mu \tag{2.22}$$

Here, dx^μ is a coordinate basis of the cotangent space, e^a is the basis of the new vector space and $e^a{}_\mu$ is the matrix representation of the isomorphism connecting the two spaces, also called the *vielbein field.* The new space is also called the *coframe* space and e^a is said to have *internal* indices while dx^μ has ordinary spacetime indices. Both sets of indices range from 1 to n, where n denotes the number of dimensions.

Also the basis vectors of the tangent space can be mapped to the dual vector space of the one just introduced

$$E_a = \tilde{e}^\mu{}_a \partial_\mu \tag{2.23}$$

Vectors from the coframe space act as linear forms on vectors from the frame space. If we want the two bases we just introduced to be dual then $\tilde{e}^\mu{}_b$ must be the inverse of the vielbein field,

$$\delta^a_b \overset{!}{=} e^a E_b = e^a{}_\mu dx^\mu e^\nu{}_b \partial_\nu = e^a{}_\mu \tilde{e}^\mu{}_b \tag{2.24}$$

The vielbein and its inverse can be used to move any spacetime tensor to the corresponding tensor product of the coframe space or its dual. Thus, if we have a tensorial object

$$T = T^{\mu_1,\dots,\mu_m}_{\nu_1,\dots,\nu_n} dx^{\nu_1} \otimes \dots \otimes dx^{\nu_n} \otimes \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_m} \tag{2.25}$$

we can re-express it in the new basis as

$$T = T^{a_1,\dots,a_m}_{b_1,\dots,b_n} e^{b_1} \otimes \dots \otimes e^{b_n} \otimes E_{a_1} \otimes \dots \otimes E_{a_m} \tag{2.26}$$

where the coefficients are now given by

$$T^{a_1,\dots,a_m}_{b_1,\dots,b_n} = e^{a_1}_{\mu_1} \dots e^{a_m}_{\mu_m} e^{\nu_1}_{b_1} \dots e^{\nu_n}_{b_n} T^{\mu_1,\dots,\mu_m}_{\nu_1,\dots,\nu_n} \tag{2.27}$$

Strictly speaking, the new object is in a different tensor space than the old one, but we will use the same symbol, albeit with a different set of indices. For most purposes, it is not even necessary to think of the coframe as living in another space. One can simply regard the vielbein as a change of basis in the cotangent space, from a coordinate base to a more general one.

Let us apply the aforementioned conversion to the metric tensor:

$$g_{ab} = e^{\mu}{}_{a} e^{\nu}{}_{b} g_{\mu\nu} \tag{2.28}$$

This object defines a scalar product in the abstract analogue of the tangent space which is compatible with the change of basis, i.e. $g_{ab}V^aV^b = g_{\mu\nu}V^{\mu}V^{\nu}$. It also consistently lowers an internal contravariant index

$$\begin{aligned} g_{ab}V^b &= e^{\mu}{}_{a} e^{\nu}{}_{b} g_{\mu\nu} e^{b}{}_{\rho} V^{\rho} \\ &= e^{\mu}{}_{a} g_{\mu\nu} V^{\nu} \\ &= e^{\mu}{}_{a} V_{\mu} \\ &= V_a \end{aligned} \tag{2.29}$$

Since the spacetime metric can be diagonalized by the vielbein (point-wise and in a smooth way), we can write

$$e^{\mu}{}_{a} e^{\nu}{}_{b} g_{\mu\nu} = \eta_{ab} \tag{2.30}$$

where η_{ab} is any fixed metric with the same signature as $g_{\mu\nu}$. A choice of vielbein which accomplishes this is called an *orthogonal vielbein*, and the coframe is called an orthogonal frame. Inverting equation (2.30) yields

$$g_{\mu\nu} = e^{a}{}_{\mu} e^{b}{}_{\nu} \eta_{ab} \tag{2.31}$$

and this means that there is a one-to-one correspondence between the metric and the vielbeins. If a fixed internal metric η_{ab} is chosen, the spacetime metric is then determined by the vielbein. One can as well use the vielbeins instead of the metric components as fundamental variables of gravity.

Of course, the vielbein has a priori more independent components that the metric. But it is not unique. Namely, if e^a is an orthogonal coframe and $\Lambda^{a}{}_{b}$ a transformation preserving η_{ab} (called a *local frame rotation*), the transformed coframe

$$\tilde{e}^a = \Lambda^{a}{}_{b} e^b \tag{2.32}$$

is also non-singular (because frame rotations are invertible) and orthogonal, as we can check by explicit calculation

$$\eta_{ab} \tilde{e}^{a}{}_{\mu} \tilde{e}^{b}{}_{\nu} = \eta_{ab} \Lambda^{a}{}_{c} \Lambda^{b}{}_{d} e^{c}{}_{\mu} e^{d}{}_{\nu} = \eta_{cd} e^{c}{}_{\mu} e^{d}{}_{\nu} = g_{\mu\nu} \tag{2.33}$$

This means that we have the same number of local degrees of freedom in the metric tensor and in the orthogonal vielbein. The metric tensor in n

dimensions has $\frac{1}{2}n(n+1)$ independent components because it is symmetric. The vielbein has a priori n^2 entries, but the group of frame rotations has $\frac{1}{2}n(n-1)$ generators. Thus, the number of independent components in the vielbein is

$$n^2 - \tfrac{1}{2}n(n-1) = \tfrac{1}{2}n(n+1) \tag{2.34}$$

the rest being gauge degrees of freedom. The further development of the theory needs to preserve this gauge symmetry. The formalism is very useful in that respect because any tensor under general coordinate transformations, when converted by vielbeins, maps to a tensor under local frame rotations.

2.3 Differential forms

One problem of ordinary tensor calculus is that it cannot be made generally covariant without introducing a metric or a connection. If we restrict ourselves to antisymmetric tensors, it is possible to set up a fully covariant calculus in absence of any additional structures. Let us see how differential forms just do that in a very simple way.

A differential p-form is a section of the p-th exterior power of the cotangent bundle of the manifold $\Lambda^{(p)}(M)$. In a local coordinate basis, it can be written as

$$A^{(p)} = \frac{1}{p!} A_{\mu_1\mu_2\ldots\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p} \tag{2.35}$$

where $\wedge$ is called the *wedge* product. It satisfies

$$A^{(p)} \wedge A^{(q)} = (-1)^{pq} A^{(q)} \wedge A^{(p)} \tag{2.36}$$

and the coefficients $A_{\mu_1\mu_2\ldots\mu_p}$ can always (without loss of generality) be taken to be totally antisymmetric. Any antisymmetric tensor with more than n indices (where n denotes the dimension of the manifold) is necessarily zero. Thus there are $n+1$ bundles of non-zero (fiber) dimension. The list starts with $\Lambda^{(0)}(M)$, the space of real functions on M. $\Lambda^{(1)}(M)$ is the cotangent bundle, which is n-dimensional, and so forth. Finally, the bundle of top-forms, $\Lambda^{(n)}(M)$ is again one-dimensional. The dimension of the intermediate powers is

$$\dim \Lambda^{(p)}(M) = \binom{n}{p} \tag{2.37}$$

Using vielbeins, we can express any differential form in a coframe basis

$$A^{(p)} = \frac{1}{p!} A_{a_1a_2\ldots a_p} e^{a_1} \wedge e^{a_2} \wedge \ldots \wedge e^{a_p} \tag{2.38}$$

and to simplify the notation, one often writes the product of coframes using the abbreviation

$$e^{a_1} \wedge e^{a_2} \wedge \ldots \wedge e^{a_p} = e^{a_1a_2\ldots a_p} \tag{2.39}$$

Exterior derivative

We can define an *exterior derivative* operator d which turns any $p-$form into a $p+1-$form. In a coordinate basis, it is defined[3] locally as

$$dA^{(p)} = \partial_\rho dx^\rho \wedge A = \frac{1}{p!}\partial_\rho A_{\mu_1\mu_2\ldots\mu_p} dx^\rho \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p} \tag{2.40}$$

and satisfies a graded Leibniz rule

$$d\left(A^{(p)} \wedge B^{(q)}\right) = dA^{(p)} \wedge B^{(q)} + (-1)^p A^{(p)} \wedge dB^{(q)} \tag{2.41}$$

Since second partial derivatives commute, applying the exterior derivative twice leads to

$$\begin{aligned} d^2 A^{(p)} &\sim \partial_\rho \partial_\sigma A_{\mu_1\mu_2\ldots\mu_p} dx^\rho \wedge dx^\sigma \wedge \ldots \\ &= \partial_\sigma \partial_\rho A_{\mu_1\mu_2\ldots\mu_p} dx^\rho \wedge dx^\sigma \wedge \ldots \\ &= -\partial_\sigma \partial_\rho A_{\mu_1\mu_2\ldots\mu_p} dx^\sigma \wedge dx^\rho \wedge \ldots \\ &\sim -d^2 A^{(p)} \end{aligned} \tag{2.42}$$

and hence

$$d^2 = 0 \tag{2.43}$$

Let us illustrate this with an example. Let the 1-form $A = A_\mu dx^\mu$ denote a $U(1)$ gauge field. If we define a corresponding 2-form $F = dA$, we obtain

$$\begin{aligned} F &= \partial_\mu A_\nu dx^\mu \wedge dx^\nu \\ &= \tfrac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu \end{aligned} \tag{2.44}$$

from which we can read off that the components of $F = \frac{1}{2}F_{\mu\nu} dx^\mu \wedge dx^\nu$ are given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ which reproduces the correct expression for an abelian field strength. The homogeneous Maxwell equations $dF = d^2 A = 0$ are a trivial consequence of equation (2.43). For the very same reason the field strength is invariant under an abelian gauge transformation $A \to A + df$.

Interior derivative

There is another derivative on differential forms which is also called the contraction operator because it contracts a vector field $V = V^\mu \partial_\mu$ with the differential p-form $A^{(p)}$ to produce a $(p-1)$-form

$$i_V A^{(p)} = \frac{1}{(p-1)!} V^{\mu_1} A_{\mu_1\mu_2\ldots\mu_p} dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p} \tag{2.45}$$

It is called a derivative because it also satisfies the graded Leibniz rule

$$i_V\left(A^{(p)} \wedge B^{(q)}\right) = i_V A^{(p)} \wedge B^{(q)} + (-1)^p A^{(p)} \wedge i_V B^{(q)} \tag{2.46}$$

[3] Of course, this can only work since $\partial_{[\rho} A_{\mu_1\mu_2\ldots\mu_p]}$ is a tensor.

The fact that the 1-forms dx^μ and the vector fields ∂_μ are dual is expressed as

$$i_\mu dx^\nu = \delta^\nu_\mu \tag{2.47}$$

where $i_\mu = i_{\partial_\mu}$. We can also express contractions in terms of the general bases e^a and E_a. If we have a vector field $V = V^a E_a$ and denote $i_{E_a} = i_a$, we have

$$i_V A^{(p)} = V^a i_a A^{(p)} \tag{2.48}$$

and the duality in (2.47) now means that

$$i_a e^b = \delta^b_a \tag{2.49}$$

Among others, the property

$$e^a \wedge i_a A^{(p)} = pA^{(p)} \tag{2.50}$$

is particularly useful.

Hodge duality

The exterior algebra of differential forms has a very nice property, already expressed in (2.37), namely that p-forms and $(n-p)$-forms have the same number of independent components. If we have a metric on our manifold (and this is the first time since our introduction of differential forms that we actually need one), we can define an isomorphism between any two bundles of the same dimension. This is called the *Hodge duality*. The Hodge dual $*A^{(p)}$ of a p-form is the $(n-p)$-form defined (in a general frame), as

$$*A^{(p)} = \frac{1}{p!} A_{a_1 a_2 \ldots a_p} * e^{a_1 a_2 \ldots a_p} \tag{2.51}$$

where

$$*e^{a_1 a_2 \ldots a_p} = \frac{|g|^{1/2}}{(n-p)!} g^{a_1 b_1} g^{a_2 b_2} \ldots g^{a_p b_p} \epsilon_{b_1 b_2 \ldots b_p b_{p+1} \ldots b_n} e^{b_{p+1} \ldots b_n} \tag{2.52}$$

where $\epsilon_{b_1 b_2 \ldots b_n}$ is the completely antisymmetric tensor in n dimensions. Although this expression may look complicated, it does a very simple thing. In an orthogonal frame, where $g_{ab} = \eta_{ab}$ and $|g| = 1$, (2.52) simplifies to

$$*e^{a_1 a_2 \ldots a_p} = \frac{1}{(n-p)!} \epsilon^{a_1 a_2 \ldots a_p}{}_{a_{p+1} \ldots a_n} e^{a_{p+1} \ldots a_n} \tag{2.53}$$

Thus, the Hodge dual simply selects all the vielbeins which are orthogonal to the original ones. In $n = 3$ Euclidean space, for example,

$$\begin{gathered} *1 = e^{123} \\ *e^1 = e^{23}, \quad *e^2 = e^{31}, \quad *e^3 = e^{12} \\ *e^{12} = e^3, \quad *e^{23} = e^1, \quad *e^{31} = e^2 \\ *e^{123} = 1 \end{gathered} \tag{2.54}$$

In this example we also see that the square of the Hodge operator is one. In general this is not the case, but almost,

$$*^2 A^{(p)} = (-1)^{p(n-p)} \mathrm{sign}\,(g)\, A^{(p)} \tag{2.55}$$

Combining the Hodge star and exterior derivatives, one can write down a large number of equations. Revisiting the example of Maxwell theory, we can now also write the inhomogeneous equations in four dimensions,

$$\nabla_\mu F^{\mu\nu} = -J^\nu \tag{2.56}$$

in terms of differential forms. We had introduced the field strength F. Let us take its dual

$$\begin{aligned} *F &= \tfrac{1}{2} F_{\mu\nu} \tfrac{1}{2} |g|^{1/2}\, g^{\mu\rho} g^{\nu\sigma} \epsilon_{\rho\sigma\lambda\gamma} dx^\lambda \wedge dx^\gamma \\ &= \tfrac{1}{4} |g|^{1/2}\, \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} dx^\rho \wedge dx^\sigma \end{aligned} \tag{2.57}$$

if we compute the derivative $d * F$, we obtain

$$d * F = \tfrac{1}{4} \epsilon_{\mu\nu\rho\sigma} \partial_\lambda \left(|g|^{1/2}\, F^{\mu\nu} \right) dx^\lambda \wedge dx^\rho \wedge dx^\sigma \tag{2.58}$$

and when acted upon with another Hodge star, after a few steps, this turns into

$$\begin{aligned} *d * F &= \tfrac{1}{4} |g|^{1/2}\, g^{\lambda\alpha} g^{\rho\beta} g^{\sigma\tau} \epsilon_{\alpha\beta\tau\gamma} \epsilon_{\mu\nu\rho\sigma} \partial_\lambda \left(|g|^{1/2}\, F^{\mu\nu} \right) dx^\gamma \\ &= \tfrac{1}{4} g^{-1} |g|^{1/2}\, \epsilon^{\lambda\xi\rho\sigma} \epsilon_{\mu\nu\rho\sigma} g_{\xi\gamma} \partial_\lambda \left(|g|^{1/2}\, F^{\mu\nu} \right) dx^\gamma \\ &= \tfrac{1}{2} |g|^{-1/2} \left(\delta^\lambda_\mu \delta^\xi_\nu - \delta^\xi_\mu \delta^\lambda_\nu \right) g_{\xi\gamma} \partial_\lambda \left(|g|^{1/2}\, F^{\mu\nu} \right) dx^\gamma \\ &= g_{\nu\gamma} |g|^{-1/2}\, \partial_\mu \left(|g|^{1/2}\, F^{\mu\nu} \right) dx^\gamma \end{aligned} \tag{2.59}$$

Now, since the covariant divergence $\nabla_\mu F^{\mu\nu}$ of an antisymmetric tensor (such as the field strength) can be written as

$$\nabla_\mu F^{\mu\nu} = |g|^{-1/2}\, \partial_\mu \left(|g|^{1/2}\, F^{\mu\nu} \right) \tag{2.60}$$

our expression $*d*F$ is just the one that is needed. Using (2.56), we can write

$$\begin{aligned} *d * F &= g_{\nu\gamma} \nabla_\mu F^{\mu\nu} dx^\gamma \\ &= -J_\mu dx^\mu \end{aligned} \tag{2.61}$$

Therefore, the inhomogeneous Maxwell equations read

$$*d * F = -J \tag{2.62}$$

And this is certainly easier to work with. For example, by using the Maxwell equations, (2.55) and $d^2 = 0$, we can easily get the conservation law

$$*d * J = - * d *^2 d * F \sim *d^2 * F = 0 \tag{2.63}$$

which corresponds to the well-known equation $\nabla_\mu J^\mu = 0$.

Integral calculus

Not only do differential forms provide us with a fully covariant differential calculus, also integrals can be defined very conveniently. Any $p-$form can be integrated over a $p-$dimensional submanifold $\mathcal{N}_p$ of M by pulling it back to a differential form on the parameter space which parametrizes $\mathcal{N}_p$ and which itself is a subset of $\mathbb{R}^p$. In the case of zero-forms, this just gives the evaluation at a specific point. In the case of one-forms, it is already more interesting. Suppose we have a curve C in our manifold which is parametrized by $\gamma : I \to M$ where $I = [t_1, t_2]$ is some interval. Then any differential one-form $\omega = \omega_\mu dx^\mu$ can be integrated along C in the following way. First the differential form ω is pulled back by γ. This reads

$$\gamma^*\omega = \omega_\mu \frac{dx^\mu}{dt} dt \tag{2.64}$$

One can then define the integral of ω along C as the integral of the pullback form over I

$$\int_C \omega = \int_I \gamma^*\omega = \int_{t_1}^{t_2} \omega_\mu \frac{dx^\mu}{dt} dt \tag{2.65}$$

If the form to be integrated is exact, i.e. $\omega = d\phi$ where ϕ is some function on M, we have by the fundamental theorem of calculus

$$\begin{aligned}\int_C d\phi &= \int_{t_1}^{t_2} \frac{\partial\phi}{\partial x^\mu}\frac{dx^\mu}{dt} dt \\ &= \int_{t_1}^{t_2} \frac{d(\phi\circ\gamma)}{dt} dt \\ &= (\phi\circ\gamma)(t_2) - (\phi\circ\gamma)(t_1) \\ &= \int_{\partial C} \phi\end{aligned} \tag{2.66}$$

where ∂C is the oriented boundary of C consisting of the two endpoints. This is but a special case of the famous *Stokes' theorem*

$$\int_{\mathcal{N}_p} d\omega^{(p-1)} = \int_{\partial\mathcal{N}_p} \omega^{(p-1)} \tag{2.67}$$

which holds for forms of any degree. The significance of this theorem for physics is that is allows to get rid of total derivatives by shifting them to the boundary.

Provided that the manifold is orientable, the form $*1$ represents the usual integration measure

$$\int_M *1 = \int_M |g|^{1/2}\, d^n x \tag{2.68}$$

If one would like to write down an action principle involving a p-form $A^{(p)}$, one must therefore turn it into a top-form in some way. This entails multiplying by an $(n-p)$ form. The simplest product one can find is

$$A^{(p)} \wedge *A^{(p)} = \frac{1}{p!} A_{a_1 a_2 \ldots a_p} A^{a_1 a_2 \ldots a_p} * 1 \tag{2.69}$$

In fact, when applied to the field strength F of a gauge theory, this yields the Yang-Mills Lagrangian

$$F \wedge *F = \tfrac{1}{2} F_{\mu\nu} F^{\mu\nu} * 1 \tag{2.70}$$

2.4 Einstein gravity in terms of forms

Let us now translate the equations of general relativity into the language of vielbeins and differential forms. As already exposed in section 2.2, the metric is replaced by the coframe e^a. What about the connection $\omega^\lambda{}_{\mu\nu}$? Equation (2.5) tells us that its last index is tensorial. This means that it can be turned into a one-form

$$\omega^\lambda{}_\mu = \omega^\lambda{}_{\mu\nu} dx^\nu \tag{2.71}$$

How can the remaining, non-tensorial indices be converted to internal ones? It easiest to do this in the special case in which the coframe field is just another coordinate basis. This means that there are coordinates ξ^a such that $e^a = d\xi^a$, and the vielbein is just the Jacobian $e^a{}_\mu = \partial\xi^a/\partial x^\mu$. In this case, the transformation can be read off from (2.5) and in terms of the vielbeins, it reads

$$\omega^a{}_b = e^a{}_\mu \omega^\mu{}_\nu e^\nu{}_b + e^a{}_\mu de^\mu{}_b \tag{2.72}$$

This will be adopted as the definition of the connection form in an arbitrary frame. It ensures that under local frame rotations we have the transformation law

$$\omega^a{}_b \to \Lambda^a{}_c \omega^c{}_d (\Lambda^{-1})^d{}_b + \Lambda^a{}_c d(\Lambda^{-1})^c{}_b \tag{2.73}$$

Therefore, we may introduce a covariant derivative with respect to frame rotations

$$D_\omega V^a = dV^a + \omega^a{}_b \wedge V^b \tag{2.74}$$

The condition of metric compatibility now reads

$$D_\omega g_{ab} = 0 \tag{2.75}$$

In an orthonormal frame, it turns into

$$D_\omega \eta_{ab} = 0 \Rightarrow \omega_{ab} = -\omega_{ba} \tag{2.76}$$

which means that the connection has values in the Lie algebra of the group of frame rotations. Together with equation (2.73) this tells us that ω is a gauge field for the group of frame rotations. This very nice structure will be exploited in the next chapters to write gravity as a gauge theory.

At this point we may also define the *curvature* and *torsion two-forms*

$$\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{2.77}$$
$$T^a = de^a + \omega^a{}_c \wedge e^c \tag{2.78}$$

Let's check by switching back to a coordinate basis that these definitions are equivalent to the previously made ones. For the curvature form, this reads

$$\begin{aligned}\Omega^\lambda{}_\sigma &= d\omega^\lambda{}_\sigma + \omega^\lambda{}_\rho \wedge \omega^\rho{}_\sigma \\ &= \tfrac{1}{2}(\partial_\mu \omega^\lambda{}_{\sigma\nu} - \partial_\nu \omega^\lambda{}_{\sigma\mu} \\ &+ \omega^\lambda{}_{\rho\mu}\omega^\rho{}_{\sigma\nu} - \omega^\lambda{}_{\rho\nu}\omega^\rho{}_{\sigma\mu}) dx^\mu \wedge dx^\nu \\ &= \tfrac{1}{2}\Omega^\lambda{}_{\sigma\mu\nu} dx^\mu \wedge dx^\nu\end{aligned} \tag{2.79}$$

so $\Omega^\lambda{}_{\sigma\mu\nu}$ has indeed the same form as the Riemann curvature tensor. For the torsion tensor, the calculation is analogous

$$\begin{aligned}T^\lambda &= ddx^\lambda + \omega^\lambda{}_\nu \wedge dx^\nu \\ &= \omega^\lambda{}_{\mu\nu} dx^\mu \wedge dx^\nu \\ &= \tfrac{1}{2}\left(\omega^\lambda{}_{\mu\nu} - \omega^\lambda{}_{\nu\mu}\right) dx^\mu \wedge dx^\nu \\ &= \tfrac{1}{2} T^\lambda{}_{\mu\nu} dx^\mu \wedge dx^\nu\end{aligned} \tag{2.80}$$

Since the curvature and torsion are tensors, the relationship between them and their internal space analogues must be

$$\Omega^a{}_b = e^a{}_\lambda \Omega^\lambda{}_\sigma e^\sigma{}_b \tag{2.81}$$
$$T^a = e^a{}_\lambda T^\lambda \tag{2.82}$$

but this could also be calculated directly from equations (2.77) and (2.78) using (2.72).

By contracting the Riemann tensor, we obtain the Ricci tensor and scalar. This is done by using the interior derivative

$$\text{Ricci 1-form}: \quad \Omega^b = i_a \Omega^{ab} \tag{2.83}$$
$$\text{Ricci scalar}: \quad \mathcal{R} = i_b \Omega^b = i_b i_a \Omega^{ab} \tag{2.84}$$

From this we can build the Einstein-Hilbert action

$$S_{\mathrm{EH}}[e] = \int *\mathcal{R} \tag{2.85}$$

By making use of the product rule for interior derivatives, we can also write the dual of the Ricci scalar as

$$\begin{aligned}*\mathcal{R} &= *1 \; i_b i_a \Omega^{ab} \\ &= (i_b i_a * 1) \wedge \Omega^{ab} \\ &= \Omega_{ab} \wedge *e^{ab}\end{aligned} \tag{2.86}$$

It is now straightforward (and much easier than in the tensor formalism) to derive the Einstein equations. The variation of the Einstein-Hilbert action reads

$$\delta \int \Omega_{ab} \wedge *e^{ab} = \int \delta\Omega_{ab} \wedge *e^{ab} + \int \Omega_{ab} \wedge \delta * e^{ab} \tag{2.87}$$

If we demand that ω is the torsion-free and metric compatible Levi-Civita connection, we have

$$\left.\begin{aligned} D_\omega g_{ab} &= 0 \\ D_\omega e^a &= 0 \end{aligned}\right\} \Rightarrow D_\omega * e^{a_1 \dots a_p} = 0 \tag{2.88}$$

and hence the first term of the variation is a total derivative, which is easy to see by using the above and $\delta\Omega_{ab} = D_\omega \delta\omega_{ab}$

$$\delta\Omega_{ab} \wedge *e^{ab} = D_\omega \delta\omega_{ab} \wedge *e^{ab} = D_\omega(\delta\omega_{ab} \wedge *e^{ab}) = d(\delta\omega_{ab} \wedge *e^{ab}) \tag{2.89}$$

The second term and hence the total variation reads

$$\delta S_{\text{EH}} = \int \delta e_c \wedge \Omega_{ab} \wedge *e^{abc} = 0 \tag{2.90}$$

and this yields the equations

$$\Omega_{ab} \wedge *e^{abc} = 0 \tag{2.91}$$

To see that these are the Einstein equations, we define the Einstein 1-form

$$G^a = \Omega^a - \tfrac{1}{2}\mathcal{R} e^a \tag{2.92}$$

and observe that its dual is proportional to equation (2.91)

$$\begin{aligned} \Omega_{ab} \wedge *e^{abc} &= \tfrac{1}{2}\Omega_{abmn} e^{mn} \wedge *e^{abc} \\ &= \tfrac{1}{2}\Omega_{abmn} e^m \wedge (g^{na} * e^{bc} + g^{nb} * e^{ca} + g^{nc} * e^{ab}) \\ &= \tfrac{1}{2}\Omega_{abmn}(g^{nb}g^{ma} - g^{na}g^{mb}) * e^c \\ &\quad + \tfrac{1}{2}\Omega_{abmn}(g^{na}g^{mc} - g^{nc}g^{ma}) * e^b \\ &\quad + \tfrac{1}{2}\Omega_{abmn}(g^{nc}g^{mb} - g^{nb}g^{mc}) * e^a \\ &= -2 * \Omega^c + \mathcal{R} * e^c \\ &= -2 * G^c \end{aligned} \tag{2.93}$$

and therefore

$$\Omega_{ab} \wedge *e^{abc} = 0 \quad \Leftrightarrow \quad G^c = 0 \tag{2.94}$$

2.5 Palatini formulation of General Relativity

We present another way to obtain the Einstein equations which will turn out to be much more useful. We make the following change: the connection is treated as a dynamical field which is metric compatible but not torsion-free, and we write down the so-called *Palatini action*

$$S_{\mathrm{Pal}}\left[\omega, e\right] = \int \Omega_{ab} \wedge *e^{ab} \tag{2.95}$$

which looks the same as the Einstein-Hilbert action. But it is now first-order in the vielbein and the spin connection. In dimensions three and four in an orthogonal frame, it reads

$$n = 3: \quad S_{\mathrm{Pal}}\left[\omega, e\right] = \int \epsilon_{abc}\, \Omega^{ab} \wedge e^c \tag{2.96}$$

$$n = 4: \quad S_{\mathrm{Pal}}\left[\omega, e\right] = \frac{1}{2} \int \epsilon_{abcd}\, \Omega^{ab} \wedge e^{cd} \tag{2.97}$$

respectively. But let us derive the equations of motion of the Palatini action in arbitrary dimension. The variation of (2.95) with respect to e_c gives rise to the Einstein equations

$$\Omega_{ab} \wedge *e^{abc} = 0 \tag{2.98}$$

The variation of the curvature can be written as

$$\int \delta\Omega_{ab} \wedge *e^{ab} = \int \delta\omega_{ab} \wedge D_\omega * e^{ab} \tag{2.99}$$

Thus, the corresponding field equation reads

$$D_\omega * e^{ab} = 0 \quad \Rightarrow \quad T^a \wedge e^b = T^b \wedge e^a \tag{2.100}$$

from which we would like to deduce $T^a = 0$. This is particularly simple in 3 dimensions since the equation reads

$$D_\omega * e^{ab} = D_\omega \epsilon^{ab}{}_c e^c = \epsilon^{ab}{}_c T^c = 0 \quad \Rightarrow \quad T^a = 0 \tag{2.101}$$

In dimension $n \geq 4$, we can contract (2.100) with i_a and obtain

$$(n-3)T^b = (i_a T^a) \wedge e^b \tag{2.102}$$

Further contraction by i_b yields

$$(2n-3) i_b T^b = (i_b i_a T^a) e^b \tag{2.103}$$

If this is substituted into equation (2.102) it implies

$$T^a = \alpha_c e^{ca}, \quad \alpha_c = \frac{2n-3}{n-3} i_c i_b T^b \tag{2.104}$$

and therefore (using (2.100) again in the last step)

$$\begin{aligned} T^a \wedge e^b = \alpha_c e^{cab} = -T^b \wedge e^a \\ \Rightarrow \quad T^a \wedge e^b = 0 \end{aligned} \tag{2.105}$$

At this point it is clear that this can only be solved if the torsion is zero, hence

$$T^a \wedge e^b = T^b \wedge e^a \quad \Rightarrow \quad T^a = 0 \tag{2.106}$$

The torsion-free connection is thus restored by the dynamics. This lucky coincidence breaks down if matter fields are introduced. In Einstein gravity, they couple to the Levi-Civita connection, which is torsion-free from the outset. In the Palatini framework, however, they couple to the spin connection and the dynamical torsion is no longer zero,

$$T \sim \frac{\delta S_{\text{Matter}}}{\delta \omega} \tag{2.107}$$

which means that the Palatini formulation is no longer equivalent to the Einstein-Hilbert action.

So far for the vacuum equations. In order to add a cosmological constant, the action (2.95) is complemented by a volume form

$$S_{\text{Pal}}[\omega, e] = \int \Omega_{ab} \wedge *e^{ab} - 2\Lambda \int *1 \tag{2.108}$$

Since $\delta * 1 = \delta e_a \wedge *e^a$, the variation with respect to ω_{ab} remains unchanged and produces $T^a = 0$ as before. On the other hand, the Einstein equations turn into

$$\Omega_{ab} \wedge *e^{abc} - 2\Lambda * e^c = 0 \tag{2.109}$$

We require that the maximally symmetric spaces, which satisfy

$$\Omega^{ab} = kl^{-2} e^{ab} \tag{2.110}$$

are a particular solution of the Einstein equations with cosmological constant. One can derive a relation between the length scale l of these spaces and the parameter Λ by plugging (2.110) into (2.109)

$$\begin{aligned} \Omega_{ab} \wedge *e^{abc} - 2\Lambda * e^c &= kl^{-2} e_{ab} \wedge *e^{abc} - 2\Lambda * e^c \\ &= \left(kl^{-2}(n-1)(n-2) - 2\Lambda\right) * e^c \\ \Rightarrow \quad \Lambda &= \tfrac{1}{2}(n-1)(n-2)kl^{-2} \end{aligned} \tag{2.111}$$

After substituting this back into (2.109), in dimension three and four we can write the Einstein equations as

$$n = 3: \quad \Omega_{ab} = kl^{-2} e^{ab} \tag{2.112}$$

$$n = 4: \quad \epsilon_{abcd} \left(\Omega^{ab} - kl^{-2} e^{ab}\right) \wedge e^c = 0 \tag{2.113}$$

In three dimensions, there are no local degrees of freedom, hence every spacetime is locally either Minkowski space, de Sitter or anti-de Sitter, depending on the sign of the cosmological constant. In four dimensions, the dynamics are more complicated, allowing for local phenomena such as gravitational waves.

In arbitrary dimension, there is another more familiar form of Einsteins equations with cosmological constant, namely

$$G^a = \Omega^a - \tfrac{1}{2}\mathcal{R}e^a = -\Lambda e^a \tag{2.114}$$

As in the tensorial language, this is not the simplest form in which to write these equations. Indeed, by contracting with i_a, we obtain

$$\mathcal{R} = \frac{2n}{n-2}\Lambda = n(n-1)kl^{-2} \tag{2.115}$$

and then, by substituting this back into (2.114),

$$\Omega^a = \frac{2}{n-2}\Lambda e^a = (n-1)kl^{-2}e^a \tag{2.116}$$

2.6 Summary

Let us summarize what was accomplished in the present chapter. We started out with conventional Einstein gravity, where the dynamical variables are the components of the metric tensor $g_{\mu\nu}$ and the Levi-Civita connection is uniquely determined by the metric.

We then trivialized the metric to η_{ab} using vielbeins as the new degrees of freedom, thereby introducing a gauge symmetry of local frame rotations preserving η_{ab}.

As a consequence, the connection 1-form ω is valued in the Lie algebra of frame rotations, its field strength given by the curvature 2-form $\Omega = d\omega + \omega\wedge\omega$. The torsion T was not put to zero by hand. Instead, the Palatini action took care of this and also yielded the Einstein equations.

3 Chern-Simons Gravity

Non-abelian gauge theories are a very successful tool in modern physics. Interactions between matter fields arise as a consequence of a local symmetry and the action principle for a given gauge group and field content is highly restricted. In this chapter, we first review some background knowledge about classical non-abelian gauge theories. We then study a very special action in three dimensions, the Chern-Simons theory which leads us to a gauge theoretical description of three dimensional gravity.

3.1 Basics of non-abelian gauge theory

A non-abelian gauge field may be described as a Lie algebra-valued one-form $A = A^a_\mu T_a dx^\mu$ where the T_a are generators of the group G, i.e. they form a basis of the Lie algebra $\mathfrak{g}$. As a connection, it transforms as

$$A \to g^{-1}Ag + g^{-1}dg \tag{3.1}$$

under a gauge transformation g. The field strength of A is[1]

$$F_A = dA + A \wedge A = dA + \tfrac{1}{2}[A, A] \tag{3.4}$$

and it transforms homogeneously

$$F_A \to g^{-1}F_A g \tag{3.5}$$

[1] The Lie bracket of two Lie algebra-valued forms X, Y is defined using the ordinary Lie bracket on the generators and the wedge product on the form parts, i.e.

$$[X, Y] = [T_a, T_b]\, X^a \wedge Y^b \tag{3.2}$$

It satisfies

$$[X^{(p)}, Y^{(q)}] = X^{(p)} \wedge Y^{(q)} - (-1)^{pq} Y^{(q)} \wedge X^{(p)} = (-1)^{pq+1}[Y^{(q)}, X^{(p)}] \tag{3.3}$$

In the case of the 1-form A, we have $[A, A] = 2A \wedge A$ and therefore the field strength can be written in the two ways given in (3.4).

as is shown by a straightforward calculation, using the fact that $dg^{-1} = -g^{-1}dgg^{-1}$

$$\begin{aligned} F_A &\to d(g^{-1}Ag + g^{-1}dg) + \left(g^{-1}Ag + g^{-1}dg\right) \wedge \left(g^{-1}Ag + g^{-1}dg\right) \\ &= dg^{-1} \wedge Ag + g^{-1}dAg - g^{-1}A \wedge dg + dg^{-1} \wedge dg \\ &\quad + g^{-1}A \wedge Ag + g^{-1}dg \wedge g^{-1}dg + g^{-1}A \wedge dg + g^{-1}dg \wedge g^{-1}Ag \\ &= g^{-1}\left(dA + A \wedge A\right) g \\ &= g^{-1}F_A g \end{aligned} \tag{3.6}$$

The gauge connection defines a covariant derivative on every associated vector bundle. In particular, for a Lie algebra valued p-form

$$V^{(p)} = \frac{1}{p!} V^a_{\mu_1,\ldots,\mu_p} T_a dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \tag{3.7}$$

which transforms in the adjoint representation of G,

$$V^{(p)} \to g^{-1}V^{(p)}g \tag{3.8}$$

the covariant derivative may be defined as

$$\begin{aligned} D_A V^{(p)} &= dV^{(p)} + \left[A, V^{(p)}\right] \\ &= dV^{(p)} + A \wedge V^{(p)} - (-1)^p V^{(p)} \wedge A \end{aligned} \tag{3.9}$$

This formula is not applicable to the gauge connection, since it is in an affine representation. But when acting on the field strength, the covariant derivative gives zero

$$\begin{aligned} D_A F_A &= dF_A + A \wedge F_A - F_A \wedge A \\ &= dA \wedge A - A \wedge dA \\ &\quad + A \wedge dA + A \wedge A \wedge A - dA \wedge A - A \wedge A \wedge A \\ &= 0 \end{aligned} \tag{3.10}$$

This fact is known as the Bianchi identity. It can also be written as $D_A^2 A = 0$, where $D_A A$ is defined as

$$D_A A = dA + A \wedge A = F_A \tag{3.11}$$

3.2 The Chern-Simons action

In three spacetime dimensions, there is a particular action given for an arbitrary gauge group G by

$$S_{\mathrm{CS}}\left[A\right] = \int tr\left(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A\right) \tag{3.12}$$

It is gauge invariant provided that $tr(XY) = \langle X, Y\rangle$ is a scalar product[2] on the Lie algebra $\mathfrak{g}$, invariant under the adjoint action of G on $\mathfrak{g}$,

$$\langle g^{-1}Xg, g^{-1}Yg\rangle = \langle X, Y\rangle \tag{3.13}$$

This invariance means that the scalar product is also invariant under the adjoint action of $\mathfrak{g}$,

$$X \to [X, Z] \tag{3.14}$$

where Z is understood to be a function (0-form). The condition of invariance reads

$$\begin{aligned} 0 = \delta\langle X, Y\rangle &= \langle \delta X, Y\rangle + \langle X, \delta Y\rangle \\ &= \langle [X, Z]\,, Y\rangle + \langle X, [Y, Z]\rangle \end{aligned} \tag{3.15}$$

On differential forms, the scalar product is defined as

$$\langle X, Y\rangle = X^a \wedge Y^b \langle T_a, T_b\rangle \tag{3.16}$$

and it inherits the graded commutativity from the wedge product,

$$\langle X^{(p)}, Y^{(q)}\rangle = (-1)^{pq}\langle Y^{(q)}, X^{(p)}\rangle \tag{3.17}$$

Using this rule and equation (3.3), the adjoint-invariance from (3.15) can be generalized to a prescription for shifting around the Lie brackets inside the scalar product. In the case of forms of degrees p, q, r, it reads

$$\langle [X^{(p)}, Y^{(q)}], Z^{(r)}\rangle = \langle X^{(p)}, [Y^{(q)}, Z^{(r)}]\rangle \tag{3.18}$$

which is easy to memorize. To verify that S_{CS} is gauge invariant, we use the adjoint invariance from (3.13). We then split the integrand of the Chern-Simons action a bit differently

$$\langle A, dA\rangle + \tfrac{1}{3}\langle A, [A, A]\rangle = \langle A, F_A\rangle - \tfrac{1}{6}\langle A, [A, A]\rangle \tag{3.19}$$

and calculate the first term

$$\begin{aligned} \langle A, F_A\rangle &\to \langle g^{-1}Ag + g^{-1}dg, g^{-1}F_A g\rangle \\ &= \langle A, F_A\rangle + \langle dgg^{-1}, F_A\rangle \end{aligned} \tag{3.20}$$

and the second one

$$\begin{aligned} \langle A, [A, A]\rangle \to \langle A, [A, A]\rangle &+ \langle dgg^{-1}, [A, A]\rangle + 2\langle A, [A, dgg^{-1}]\rangle \\ &+ 2\langle dgg^{-1}, [A, dgg^{-1}]\rangle + \langle A, [dgg^{-1}, dgg^{-1}]\rangle \\ &+ \langle dgg^{-1}, [dgg^{-1}, dgg^{-1}]\rangle \end{aligned} \tag{3.21}$$

[2] The symbol tr is reminiscent of the Killing form, which exists and is invariant for any Lie algebra, but only non-degenerate for the semi-simple ones. We shall consider the most general invariant non-degenerate bilinear form.

Together, they add up to

$$\begin{aligned}\langle A, dA\rangle + \tfrac{1}{3}\langle A, [A, A]\rangle \to \langle A, dA\rangle &+ \tfrac{1}{3}\langle A, [A, A]\rangle \\ &- \tfrac{1}{3}\langle dgg^{-1}, dgg^{-1} \wedge dgg^{-1}\rangle + d\langle A, dgg^{-1}\rangle\end{aligned} \tag{3.22}$$

This means that the Chern-Simons action is gauge invariant up to a topological and a surface term

$$S_{\mathrm{CS}}[A] \to S_{\mathrm{CS}}[A] - \frac{1}{3}\int tr(g^{-1}dg)^3 + \oint tr(A \wedge dgg^{-1}) \tag{3.23}$$

Field equations

The field equations of Chern-Simons theory turn out to be of a very simple form. Let us calculate them. The first term in the variation is

$$\begin{aligned}\delta\,\langle A, dA\rangle &= \langle \delta A, dA\rangle + \langle A, d\delta A\rangle \\ &= 2\langle \delta A, dA\rangle + d\langle \delta A, A\rangle\end{aligned} \tag{3.24}$$

where partial integration and the graded symmetry of the scalar product were used in the second step. By making use of the adjoint invariance, we find that the second term has the form

$$\begin{aligned}\delta\,\langle A, A \wedge A\rangle &= \langle \delta A, A \wedge A\rangle + \langle A, \delta A \wedge A\rangle + \langle A, A \wedge \delta A\rangle \\ &= \langle \delta A, A \wedge A\rangle + \langle A, [A, \delta A]\rangle \\ &= 3\langle \delta A, A \wedge A\rangle\end{aligned} \tag{3.25}$$

The variation of the action thus reads

$$\delta S_{\mathrm{CS}} = 2\int \langle \delta A, F_A\rangle + \oint \langle \delta A, A\rangle \tag{3.26}$$

The boundary term has to vanish and the resulting field equations are

$$F_A = 0 \tag{3.27}$$

provided that the scalar product is non-degenerate. This means that classical solutions of Chern-Simons theory are the flat G-connections. This also means that whenever the manifold we are working on is trivial enough, every solution is *pure gauge*, i.e. a gauge transform of the trivial solution $A = 0$,

$$F_A = 0 \quad \Rightarrow \quad A = g^{-1}dg \tag{3.28}$$

We shall return to this peculiar feature in due time. For the moment, let us see how Chern-Simons theory leads to three dimensional gravity.

3.3 Chern-Simons formulation of 3d gravity

In this section, we want to formulate 2+1-dimensional gravity with vanishing cosmological constant as a Chern-Simons gauge theory. This construction goes back to Witten [9]. What is the right gauge group? In the Palatini formulation of GR, we had the spin connection ω and the triad e as our fundamental variables. In dimension three, each of them has three independent components, and we know that ω has values in the Lie algebra of the Lorentz group $H = SO(2,1)$. This suggests to use the group $G = ISO(2,1)$ and parametrize the gauge field as

$$A = \omega + e = \omega^i J_i + e^i P_i \tag{3.29}$$

where P_i are the translations and J_i contain the two Lorentz boosts and one spatial rotation. The field ω^i is but a different parametrization of the spin connection degrees of freedom, namely

$$\omega^i = \tfrac{1}{2}\epsilon^{ijk}\omega_{jk} \tag{3.30}$$

The generators satisfy the commutation relations

$$\begin{aligned} [J_i, J_j] &= \epsilon_{ijk} J^k \\ [J_i, P_j] &= \epsilon_{ijk} P^k \\ [P_i, P_j] &= 0 \end{aligned} \tag{3.31}$$

where indices are raised and lowered with the metric $\eta = \mathrm{diag}\,(-1,1,1)$ and we choose $\epsilon_{012} = 1$. From these relations, one can calculate the field strength

$$\begin{aligned} F_A &= dA + \tfrac{1}{2}\,[A, A] \\ &= \left(d\omega + \tfrac{1}{2}\,[\omega,\omega]\right) + (de + [\omega, e]) \\ &= \Omega + T \end{aligned} \tag{3.32}$$

where

$$\Omega = \Omega^i J_i = \left(d\omega^i + \tfrac{1}{2}\epsilon^i{}_{jk}\omega^j \wedge \omega^k\right) J_i \tag{3.33}$$

$$T = T^i P_i = \left(de^i + \epsilon^i{}_{jk}\omega^j \wedge e^k\right) P_i \tag{3.34}$$

It can be guessed that these forms are related to the curvature and torsion. Indeed, if equation (3.30) is inverted, it reads

$$\omega^i{}_j = -\epsilon^i{}_{jk}\omega^k \tag{3.35}$$

and if this is injected into equation (2.77), the curvature form is expressed as

$$\begin{aligned} \Omega^i{}_j &= d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j \\ &= -\epsilon^i{}_{jk} d\omega_k + \epsilon^i{}_{kl}\epsilon^k{}_{jm}\omega^l \wedge \omega^m \\ &= -\epsilon^i{}_{jk} d\omega_k + \omega^i \wedge \omega_j \end{aligned} \tag{3.36}$$

on the other hand, from equation (3.33) one has

$$\begin{aligned} -\epsilon^i{}_{jk}\Omega^k &= -\epsilon^i{}_{jk}d\omega_k - \tfrac{1}{2}\epsilon^i{}_{jk}\epsilon^{klm}\omega_l \wedge \omega_m \\ &= -\epsilon^i{}_{jk}d\omega_k + \omega^i \wedge \omega_j \end{aligned} \tag{3.37}$$

Therefore, Ω^i and the curvature form have the same relationship as ω^i and the spin connection, namely

$$\Omega^i{}_j = -\epsilon^i{}_{jk}\Omega_k \quad \Longleftrightarrow \quad \Omega^i = \tfrac{1}{2}\epsilon^{ijk}\Omega_{jk} \tag{3.38}$$

Moreover, the T part in F_A looks like and turns out to be the torsion two-form as defined in equation (2.78)

$$\begin{aligned} T^i &= de^i + \epsilon^i{}_{jk}\omega^j \wedge e^k \\ &= de^i + \tfrac{1}{2}\epsilon^i{}_{jk}\epsilon^j{}_{lm}\omega^{lm} \wedge e^k \\ &= de^i + \omega^i{}_k \wedge e^k \end{aligned} \tag{3.39}$$

The whole theory does not make any sense unless there is an invariant non-degenerate bilinear form on the Lie algebra of $ISO(2,1)$. It can be shown[3] that

$$\langle J_i, P_j \rangle = \eta_{ij}, \quad \langle J_i, J_j \rangle = \langle P_i, P_j \rangle = 0 \tag{3.40}$$

solves (3.15) and is non-singular. Therefore when we plug this scalar product into the action, the field equations read $F_A = 0$ or, equivalently

$$\Omega = 0 \tag{3.41}$$

$$T = 0 \tag{3.42}$$

These are the same equations as those we get from the Palatini action. The field equation would have been identical had we used a different scalar product, and we will see in the next section that there are more scalar products to be chosen. But when the one from equation (3.40) is used, the Chern-Simons action takes a special form. Let's see what it looks like. The first term contributes

$$\begin{aligned} \langle A, dA \rangle &= \langle e, d\omega \rangle + \langle \omega, de \rangle \\ &= 2\langle e, d\omega \rangle - d\langle \omega, e \rangle \\ &= 2e^i \wedge d\omega_i - d(\omega^i \wedge e_i) \end{aligned} \tag{3.43}$$

while the second one adds the following piece

$$\begin{aligned} \tfrac{1}{3}\langle A, [A, A] \rangle &= \tfrac{1}{3}\langle e, [\omega, \omega] \rangle + \tfrac{1}{3}\langle \omega, [\omega, e] \rangle + \tfrac{1}{3}\langle \omega, [e, \omega] \rangle \\ &= \langle e, [\omega, \omega] \rangle \\ &= e^i \wedge \left(\epsilon_{ijk}\omega^j \wedge \omega^k \right) \end{aligned} \tag{3.44}$$

[3] In the next section we will give a general argument for this.

	$\Lambda < 0$	$\Lambda = 0$	$\Lambda > 0$
Riemannian	EAdS $SO(3,1) \big/ SO(3)$	Euclidean $ISO(3) \big/ SO(3)$	EAdS $SO(4) \big/ SO(3)$
Lorentzian	AdS $SO(2,2) \big/ SO(2,1)$	Minkowski $ISO(2,1) \big/ SO(2,1)$	dS $SO(3,1) \big/ SO(2,1)$

Table 3.1: Homogeneous spacetime models in dimension three

Together, they add up to the Palatini action modulo a boundary term

$$\begin{aligned} S_{\text{CS}}\left[e+\omega\right] + \oint \omega^i \wedge e_i &= 2\int e^i \wedge \Omega_i \\ &= \int \epsilon_{ijk}\Omega^{ij} \wedge e^k = S_{\text{Pal}}\left[e,\omega\right] \end{aligned} \tag{3.45}$$

3.4 Homogeneous spaces

We would like to generalize the Chern-Simons formulation of GR also to cases with non-zero cosmological constant, with either Riemannian (positive definite) or Lorentzian signature. All of the six models are summarized in table 3.4. Each of them is a homogeneous space

$$M = {}^{G}\big/_{H} \tag{3.46}$$

with a symmetry group G acting transitively on M and the stabilizer subgroup of a point, denoted by H. The general idea is to use a connection valued in the Lie algebra $\mathfrak{g}$ of G, where the part in the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ will be the spin connection and the part in $\mathfrak{g} \setminus \mathfrak{h}$ will be interpreted as the vielbein. The splitting will thus look essentially the same as before

$$A = \omega + \tfrac{1}{l}e \tag{3.47}$$

where the only difference to the previous discussion is the length scale l that is introduced (and was set to 1 up to now). We already have the flat Lorentzian model (Minkowski space), and the corresponding Riemannian model is completely analogous except for some sign changes. In this section, we will cover all the six models at the same time. However, special care has to be taken regarding the scalar products on the Lie algebras.

There are some definitions to be done. Let k be the sign of the cosmological constant Λ. For the cases with $k \neq 0$, we fix the metric which is invariant under G to be

$$\eta_{ab} = \begin{pmatrix} \eta' & & & \\ & 1 & & \\ & & 1 & \\ & & & k \end{pmatrix} \tag{3.48}$$

where the indices $a, b, ...$ range from 1 to 4 in the Riemannian models and from 0 to 3 in the Lorentzian ones. The upper 3×3 block denoted by η'_{ij} is invariant under H and determines the signature of the space, with $\eta' = 1$ for the Riemannian and $\eta' = -1$ for the Lorentzian cases. Indices $i, j, ...$ are meant to go from 1 to 3 in the Riemannian cases and from 0 to 2 in the Lorentzian ones. Please note the way minus signs of the metric are chosen. As an example, consider the case when $G = SO(3,1)$. For the Riemannian anti-de Sitter (or hyperbolic) model the metric reads $\eta_{ab} = diag(1,1,1,-1)$ and for the Lorentzian de Sitter model $\eta_{ab} = diag(-1,1,1,1)$. In any case, the determinant of the full metric is

$$\eta = \eta' k \tag{3.49}$$

and we use this parameter also for when $k = 0$ (it is zero then), even if we don't have an ambient metric but only the Riemannian or Lorentzian 3-metric $\eta'_{ij} = diag(\eta', 1, 1)$ in those cases. This parameter allows us to choose a basis[4] for which we have

$$\begin{aligned} [J_i, J_j] &= \epsilon_{ijk} J^k \\ [J_i, P_j] &= \epsilon_{ijk} P^k \\ [P_i, P_j] &= \eta \epsilon_{ijk} J^k \end{aligned} \tag{3.50}$$

where indices are raised and lowered by η'_{ij} and $\epsilon_{123} = \epsilon_{012} = 1$. This and the following holds for all six models to be studied. In the field strength,

$$\begin{aligned} F_A &= dA + \tfrac{1}{2}[A, A] \\ &= \left(d\omega + \tfrac{1}{2}[\omega,\omega] + \tfrac{1}{2} l^{-2} [e, e]\right) + \tfrac{1}{l}(de + [\omega, e]) \\ &= \hat{F} + \tfrac{1}{l} T \end{aligned} \tag{3.51}$$

the non-vanishing commutator of the transvections is responsible for a new term in the $\mathfrak{h}$-valued part $\hat{F}$, sometimes called the corrected curvature, which reads

$$\begin{aligned} \hat{F}^i &= d\omega^i + \tfrac{1}{2}\epsilon^{ijk}\omega_j \wedge \omega_k + \tfrac{1}{2}\eta l^{-2} \epsilon^{ijk} e_j \wedge e_k \\ &= \Omega^i + \tfrac{1}{2}\eta l^{-2}\epsilon^{ijk} e_j \wedge e_k \end{aligned} \tag{3.52}$$

The other piece is the torsion, as before given by

$$T^i = de^i + \epsilon^i{}_{jk} \omega^j \wedge e^k \tag{3.53}$$

By using the relation $\omega^i{}_j = -\epsilon^i{}_{jk}\omega^k$ one finds

$$\hat{F}^i = -\tfrac{1}{2}\eta' \epsilon^{ijk}\left(\Omega_{jk} - k l^{-2} e_j \wedge e_k\right) \tag{3.54}$$

$$T^i = de^i + \omega^i{}_j \wedge e^j \tag{3.55}$$

[4]See appendix A.1 for reference.

Bilinear forms

At this point we must find an invariant scalar product. In terms of the generators, there are three equations to be solved

$$\langle [J_i, J_k], J_l\rangle = \langle J_i, [J_k, J_l]\rangle \tag{3.56}$$
$$\langle [J_i, J_k], P_l\rangle = \langle J_i, [J_k, P_l]\rangle \tag{3.57}$$
$$\langle [J_i, P_k], P_l\rangle = \langle J_i, [P_k, P_l]\rangle \tag{3.58}$$

By using the commutation relations (3.50), one can write the first as

$$\epsilon_{ikm}\langle J^m, J_l\rangle = \epsilon_{klm}\langle J_i, J^m\rangle \tag{3.59}$$

After multiplying by ${\epsilon^{ik}}_n$ and contracting the indices, we get

$$2\langle J_n, J_l\rangle = -\langle J_l, J_n\rangle + \eta_{nl}\langle J_i, J^i\rangle \tag{3.60}$$

The same manipulations, when applied to the other two equations, yield

$$2\langle J_n, P_l\rangle = -\langle J_l, P_n\rangle + \eta_{nl}\langle J_i, P^i\rangle \tag{3.61}$$
$$2\langle P_n, P_l\rangle = -\eta\langle J_l, J_n\rangle + \eta\eta_{nl}\langle J_i, J^i\rangle \tag{3.62}$$

Equation (3.60) means that the scalar product of two rotations is proportional to the metric

$$3\langle J_i, J_k\rangle = \eta_{ik}\langle J_l, J^l\rangle \quad \Rightarrow \quad \langle J_i, J_k\rangle = \alpha\eta_{ik} \tag{3.63}$$

where α is an arbitrary real constant. Similarly, the second equation implies

$$\langle J_i, P_k\rangle = \beta\eta_{ik} \tag{3.64}$$

with another independent constant β. The remaining scalar product is determined by these choices

$$\begin{aligned} 2\langle P_i, P_k\rangle &= -\eta\alpha\eta_{ik} + \eta\eta_{ik}\alpha\eta^l_l \\ &= 2\alpha\eta\eta_{ik} \\ \Rightarrow \quad \langle P_i, P_k\rangle &= \alpha\eta\eta_{ik} \end{aligned} \tag{3.65}$$

Therefore for any choice of α and β, the equations

$$\langle J_i, J_k\rangle = \alpha\eta_{ik}, \quad \langle J_i, P_j\rangle = \beta\eta_{ij}, \quad \langle P_i, P_j\rangle = \alpha\eta\eta_{ik} \tag{3.66}$$

determine an invariant scalar product on any of the Lie algebras $\mathfrak{so}(4)$, $\mathfrak{so}(2,2)$, $\mathfrak{so}(3,1)$ and $\mathfrak{iso}(3)$, $\mathfrak{iso}(2,1)$. It is non-degenerate if the determinant is non-zero

$$D(\alpha,\beta) = \eta'^2\left(\eta\alpha^2 - \beta^2\right)^3 \neq 0 \tag{3.67}$$

For $\eta = 0$, corresponding to the flat models, the non-vanishing determinant requires $\beta \neq 0$, not putting any constraint on α. Thus we find, besides the scalar product we used in the previous section (which corresponds to $\alpha = 0$ and $\beta = 1$), a one-parameter family of scalar products[5]. The same is true if $\eta \neq 0$. When $\eta < 1$ there is no restriction whatsoever on the parameters, while if $\eta > 1$ the scalar product is degenerate for $\beta = \pm\alpha$.

[5]The second parameter can be thought of as an overall normalization.

Actions

Given any non-degenerate invariant scalar product, the field equations contained in $F_A = 0$ are again equivalent to the Einstein equations with vanishing torsion

$$\hat{F}^i = 0 \quad \Rightarrow \quad \Omega^{ij} = kl^{-2} e^i \wedge e^j \tag{3.68}$$
$$T^i = de^i + \omega^i{}_j \wedge e^j = 0 \tag{3.69}$$

But the form of the action depends on the particular scalar product used. Indeed, if we split the most general bilinear form from equation (3.66)

$$\langle\ ,\ \rangle = \alpha\langle\ ,\ \rangle_1 + \beta\langle\ ,\ \rangle_2 \tag{3.70}$$

such that the two parts are obviously given by

$$\langle J_i, J_k\rangle_1 = \eta_{ik}, \qquad \langle J_i, P_j\rangle_1 = 0, \qquad \langle P_i, P_j\rangle_1 = \eta\eta_{ik} \tag{3.71}$$
$$\langle J_i, J_k\rangle_2 = 0, \qquad \langle J_i, P_j\rangle_2 = \eta_{ij}, \qquad \langle P_i, P_j\rangle_2 = 0 \tag{3.72}$$

the Chern-Simons action also splits into two parts

$$S_{\mathrm{CS}}^{(\alpha,\beta)} = \alpha S_{\mathrm{CS}}^{(1)} + \beta S_{\mathrm{CS}}^{(2)} \tag{3.73}$$

The integrand of the first action reads

$$\begin{aligned}\langle A, dA\rangle_1 + \tfrac{1}{3}\langle A, [A,A]\rangle_1 &= l^{-2}\langle e, de\rangle_1 + \tfrac{2}{3}l^{-2}\langle e, [\omega, e]\rangle_1 + \tfrac{1}{3}l^{-2}\langle \omega, [e,e]\rangle_1 \\ &\quad + \langle \omega, d\omega\rangle_1 + \tfrac{1}{3}\langle \omega, [\omega,\omega]\rangle_1 \\ &= l^{-2}\langle e, T\rangle_1 + \langle \omega, d\omega\rangle_1 + \tfrac{1}{3}\langle \omega, [\omega,\omega]\rangle_1\end{aligned} \tag{3.74}$$

and thus the action pertaining to the first bilinear form consists of a Chern-Simons term in ω and a torsional term

$$S_{\mathrm{CS}}^{(1)}[A] = S_{\mathrm{CS}}^{(1)}[\omega] + l^{-2}\int \langle e, T\rangle_1 \tag{3.75}$$

The action $S_{\mathrm{CS}}^{(2)}$ contains the following pieces

$$\begin{aligned}\langle A, dA\rangle_2 + \tfrac{1}{3}\langle A, [A,A]\rangle_2 &= \tfrac{1}{l}\langle e, d\omega\rangle_2 + \tfrac{1}{l}\langle \omega, de\rangle_2 \\ &\quad + \tfrac{1}{3}l^{-3}\langle e, [e,e]\rangle_2 + \tfrac{1}{l}\langle e, [\omega,\omega]\rangle_2 \\ &= \tfrac{2}{l}\langle e, \Omega\rangle_2 + \tfrac{1}{3}l^{-3}\langle e, [e,e]\rangle_2 - \tfrac{1}{l}d\langle \omega, e\rangle_2\end{aligned} \tag{3.76}$$

and this gives rise to the Palatini action with cosmological constant $\Lambda = kl^{-2}$

$$\begin{aligned}S_{\mathrm{CS}}^{(2)}[A] &= \frac{1}{l}\int \left(2e^i \wedge \Omega_i + \tfrac{1}{3}\eta l^{-2}\epsilon_{ijk}e^{ijk}\right) \\ &= -\frac{\eta'}{l}\int \epsilon_{ijk}\left(\Omega^{ij}\wedge e^k - \tfrac{1}{3}kl^{-2}e^{ijk}\right) \\ &= -\frac{\eta'}{l}S_{\mathrm{Pal}}[e,\omega]\end{aligned} \tag{3.77}$$

where the occurring boundary term has been left away. By the way, this illuminates why β cannot be zero in the flat models. If this were the case, the action would only consist of the term $S^{(1)}_{\mathrm{CS}}[\omega]$ (the second term in the first action would be absent because $[\mathfrak{p}, \mathfrak{p}] = 0$), which contains only the spin connection. Thus, the field equation would not involve the triad.

There is another interesting property of the Lie algebras we are dealing with. The involution map

$$i : \mathfrak{g} \to \mathfrak{g}, \quad A \mapsto \bar{A} \tag{3.78}$$

defined by

$$J_i \mapsto J_i, \quad P_i \mapsto -P_i \tag{3.79}$$

leaves invariant the commutation relations given in (3.50). Moreover, the scalar product $\langle\ ,\ \rangle_1$ does not change under this map, while $\langle\ ,\ \rangle_2$ changes sign. This means that

$$S^{(\alpha,\beta)}_{\mathrm{CS}}[\bar{A}] = \alpha S^{(1)}_{\mathrm{CS}}[A] - \beta S^{(2)}_{\mathrm{CS}}[A] \tag{3.80}$$

This behavior under a change of sign of e is also obvious by looking at how many powers of e are occurring in the different terms. The first action has only even powers while the second one involves first and third powers of e. The Palatini action can therefore be extracted from the antisymmetric part with respect to the involution whenever $\beta \neq 0$, i.e.

$$S_{\mathrm{Pal}}[e, \omega] = -\frac{\eta' l}{2\beta} \left(S^{(\alpha,\beta)}_{\mathrm{CS}}[A] - S^{(\alpha,\beta)}_{\mathrm{CS}}[\bar{A}] \right) \tag{3.81}$$

3.5 Gauge transformations

It is remarkable that the Palatini action of three dimensional gravity can be derived from a gauge theory. But by combining the vielbein and the spin connection into one single gauge field, we have significantly enlarged the group of symmetries. While the subgroup of local frame rotations is still there, comprising the three J_i out of six generators, there are three transvections or translations that come into play. It has already been remarked in one of the previous sections that any solution of the field equations

$$F_A = 0 \tag{3.82}$$

is a flat connection, and on a manifold which is of sufficiently trivial topology[6], it can related to the trivial solution $A = 0$ by a gauge transformation, or put differently

$$A = g^{-1} dg \tag{3.83}$$

[6]The exact condition is that the fundamental group $\pi_1(M)$ be trivial. This applies to all of our homogeneous spaces, except for AdS space, which is endued with closed timelike curves and has topology $\mathbb{R}^2 \times S^1$.

for some $g \in G$. This means that we cannot restrict our vielbeins to be non-degenerate. While this condition is preserved by the local frame rotations and therefore constitutes a sensible restriction for conventional gravity, it is not a natural thing to do in a gauge theory.

Let us first consider small gauge transformations, that is we pick $g = \mathbb{1} + u$, where

$$u = \tau + \tfrac{1}{l}\rho = \tau^i J_i + \tfrac{1}{l}\rho^i P_i \tag{3.84}$$

Under the transformation $A \to g^{-1}Ag + g^{-1}dg$, the variation of the gauge field can be expressed as

$$\delta A = du + [A, u] = D_A u \tag{3.85}$$

and using the commutation relations, we can derive

$$\delta e = d\rho + [e, \tau] + [\omega, \rho] \tag{3.86}$$

$$\delta\omega = d\tau + [e, \rho] + [\omega, \tau] \tag{3.87}$$

or, in components

$$\delta e^i = d\rho^i + \epsilon^{ijk} e_j \tau_k + \epsilon^{ijk}\omega_j \rho_k \tag{3.88}$$

$$\delta\omega^i = d\tau^i + \eta\epsilon^{ijk} e_j \rho_k + \epsilon^{ijk}\omega_j \tau_k \tag{3.89}$$

Thus, if $\rho = 0$, using the definition $\tau^{ij} = -\epsilon^{ijk}\tau_k$, we find

$$\delta e^i = -\tau^i{}_j e^j \tag{3.90}$$

$$\delta\omega^i{}_j = -\epsilon^i{}_{jk}\delta\omega^k = d\tau^i{}_j + \omega^i{}_k \tau^k{}_j - \tau^i{}_k \omega^k{}_j \tag{3.91}$$

which corresponds to a local frame rotation $e^i \to \Lambda^i{}_j e^j$ with $\Lambda^i{}_j = \delta^i{}_j - \tau^i{}_j$. Thus, the subgroup H generated by the J_i *is* the group of local frame rotations, as expected.

Extended gauge transformations

In contrast to small gauge transformations, for extended ones the variations of e and ω are in general difficult to calculate. There are however some general results that can be derived. Let $g = \exp(u)$, where $u = \tau + \frac{1}{l}\rho$ is a Lie algebra-valued parameter function as before (but now not necessarily small). Then, the transformation of A, which can be split into three parts

$$A = \omega + \tfrac{1}{l}e \quad \to \quad g^{-1}\omega g + \tfrac{1}{l}g^{-1}eg + g^{-1}dg \tag{3.92}$$

can be given in terms of Hadamard's Lemma. The first two terms read

$$g^{-1}\omega g = e^{-u}\omega e^u = \sum_{m=0}^{\infty} \frac{1}{m!}\,[-u, \omega]_m \tag{3.93}$$

$$g^{-1}eg = e^{-u}ee^u = \sum_{m=0}^{\infty} \frac{1}{m!}\,[-u, e]_m \tag{3.94}$$

where $[\ ,\ ]_m$ is defined recursively as

$$[X,Y]_m = \big[X,[X,Y]_{m-1}\big]\,, \quad [X,Y]_0 = Y \tag{3.95}$$

The term $g^{-1}dg$ is a bit more tricky. One can use the integral representation, and then again Hadamard's Lemma, to derive

$$\begin{aligned} g^{-1}dg &= \int_0^1 d\xi\, e^{-\xi u} du\, e^{\xi u} \\ &= \int_0^1 d\xi \sum_{m=0}^{\infty} \frac{1}{m!}\,[-\xi u, du]_m \\ &= \sum_{m=0}^{\infty} \frac{1}{m!}\,[-u, du]_m \int_0^1 d\xi\, \xi^m \\ &= \sum_{m=0}^{\infty} \frac{1}{(m+1)!}\,[-u, du]_m \end{aligned} \tag{3.96}$$

Applying these formulæ can produce very complicated expressions. One simple result however can be derived very easily, namely the fact that in the case where $\eta = 0$ (Poincaré gauge theory), translations leave the spin connection invariant. This is derived as follows (we set $l = 1$): Since for translations $\tau = 0$ and $[\mathfrak{p},\mathfrak{p}] = 0$, the series given in (3.93), (3.94) and (3.96) each only have a finite number of terms

$$\begin{aligned} [-\rho,e]_0 = e,\ [-\rho,e]_1 = [-\rho,e] = 0 \quad &\Rightarrow \quad [-\rho,e]_m = 0,\ m \geq 1 \\ [-\rho,\omega]_0 = \omega,\ [-\rho,\omega]_1 = [-\rho,\omega] \in \mathfrak{p} \quad &\Rightarrow \quad [-\rho,\omega]_m = 0,\ m \geq 2 \\ [-\rho,d\rho]_0 = d\rho,\ [-\rho,d\rho]_1 = [-\rho,d\rho] = 0 \quad &\Rightarrow \quad [-\rho,d\rho]_m = 0,\ m \geq 1 \end{aligned} \tag{3.97}$$

and hence

$$g^{-1}eg = e, \quad g^{-1}\omega g = \omega + [\omega,\rho]\,, \quad g^{-1}dg = d\rho \tag{3.98}$$

By rearranging the $\mathfrak{h}$ and $\mathfrak{p}$ components, we can read off that

$$\begin{aligned} e &\to e + D_\omega \rho \\ \omega &\to \omega \end{aligned} \tag{3.99}$$

This is of course true for infinitesimal translations, but it holds also for arbitrary parameters.

Gauge transformations and diffeomorphisms

On-shell, a large class of gauge transformations can be interpreted as small diffeomorphisms. Under a diffeomorphism generated by the vector field V, the variation of the gauge field is given by its Lie derivative

$$\delta A = \mathcal{L}_V A = i_V dA + d i_V A \tag{3.100}$$

By using the field strength $F_A = dA + \frac{1}{2}[A, A]$, one can establish the identity

$$\begin{aligned} \delta A &= i_V dA + d i_V A \\ &= i_V F_A - \tfrac{1}{2} i_V [A, A] + d i_V A \\ &= i_V F_A + d i_V A + [A, i_V A] \\ &= i_V F_A + D_A i_V A \end{aligned} \tag{3.101}$$

On-shell, where $F_A = 0$, this diffeomorphism is obtained from the gauge transformation $u = i_V A$. This means that whenever we have a field configuration which has a non-degenerate vielbein, this is certainly preserved by the diffeomorphism, and hence by any gauge transformation which is of the form given above.

3.6 Gauge transformations in the $SO(4)$ model

To gain a deeper understanding of the relevance of gauge transformations in Chern-Simons gravity, we look at the Riemannian model with positive cosmological constant, which is nothing but the three-sphere

$$SO(4) \Big/ SO(3) \cong S^3 \tag{3.102}$$

Calculations are simpler than in the other models for a number of reasons. First of all, we can exploit the isomorphisms

$$SO(4) \cong SU(2) \times SU(2)/\mathbb{Z}_2 \tag{3.103}$$

and

$$S^3 \cong SU(2) \tag{3.104}$$

to think of gauge transformations as maps from the sphere onto itself. Secondly, using some of the simple coordinate patches which are known on the sphere, quantities like the vielbeins, metrics and winding numbers resulting from these maps are comparatively easy to compute.

Preparatory calculations

Let us make things more specific. First, we define the covering map

$$\begin{aligned} SU(2) \times SU(2) &\to SO(4) \\ (g, h) &\mapsto R(g, h) \end{aligned} \tag{3.105}$$

If the two $SU(2)$ transformations are parametrized by

$$g = g_4 \mathbb{1}_2 + i g_i \sigma_i, \quad h = h_4 \mathbb{1}_2 + i h_i \sigma_i \tag{3.106}$$

with $\delta_{ab} g_a g_b = \delta_{ab} h_a h_b = 1$ and σ_i the three Pauli matrices, then the corresponding element of $SO(4)$ is given by[7]

$$R(g,h) = \left(h_4 \mathbb{1}_4 + h_i J_i^+\right)\left(g_4 \mathbb{1}_4 + g_i J_i^-\right) = \hat{h}\tilde{g} \tag{3.107}$$

where the $J_i^\pm$ are simply another set of generators related to the previous ones by

$$J_i^\pm = J_i \pm P_i \tag{3.108}$$

They express the Lie algebra isomorphism $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and therefore satisfy the commutation relations

$$\begin{aligned} \left[J_i^\pm, J_k^\pm\right] &= 2\epsilon_{ikl} J_l^\pm \\ \left[J_i^+, J_i^-\right] &= 0 \end{aligned} \tag{3.109}$$

For the practical calculations we will be doing shortly, the following results are also useful

$$\begin{aligned} J_i^\pm J_k^\pm &= -\delta_{ik} \mathbb{1}_4 + \epsilon_{ikl} J_l^\pm \\ J_i^\pm J_k^\pm J_l^\pm &= -\epsilon_{ikl} \mathbb{1}_4 - \delta_{ik} J_l^\pm - \delta_{kl} J_i^\pm + \delta_{il} J_k^\pm \end{aligned} \tag{3.110}$$

and they are the same as in $SU(2)$. These relations imply, among other things, that the two maps $SU(2) \to SO(4)$ given by

$$g \mapsto R(g, \mathbb{1}_2) = \tilde{g}, \quad h \mapsto R(\mathbb{1}_2, h) = \hat{h} \tag{3.111}$$

are representations of $SU(2)$, such that $\widetilde{g_1 g_2} = \tilde{g}_1 \tilde{g}_2$ and so forth.

The gauge field can be split along the two commuting directions

$$A = A^+ + A^- = A^{+i} J_i^+ + A^{-i} J_i^- \tag{3.112}$$

In this splitting the field strength is linear

$$F_A = F_{A^+} + F_{A^-} \tag{3.113}$$

and the relation to the vielbein and spin connection (we set $l = 1$) is given by

$$\begin{aligned} e_i &= A_i^+ - A_i^- \\ \omega_i &= A_i^+ + A_i^- \end{aligned} \tag{3.114}$$

Now suppose that the gauge field is given by a pure gauge transformation, i.e. $A = R(g,h)^{-1} dR(g,h)$. Since the two factors in R commute, things simplify considerably

$$\begin{aligned} R(g,h)^{-1} dR(g,h) &= (\hat{h}\tilde{g})^{-1} d(\hat{h}\tilde{g}) \\ &= \tilde{g}^{-1}\hat{h}^{-1}\left(d(\hat{h})\tilde{g} + \hat{h} d\tilde{g}\right) \\ &= \hat{h}^{-1} d\hat{h} + \tilde{g}^{-1} d\tilde{g} \end{aligned} \tag{3.115}$$

[7] Please have a look at appendix A.2 for a detailed derivation of this result.

meaning that the two components of the gauge field are given by the individual parts of the gauge transformation as

$$A^+ = \hat{h}^{-1} d\hat{h}, \quad A^- = \tilde{g}^{-1} d\tilde{g} \tag{3.116}$$

If this field is subjected to another gauge transformation,

$$A \to R(r,s)^{-1} A R(r,s) + R(r,s)^{-1} dR(r,s) \tag{3.117}$$

the result is again a pure gauge, where the relevant transformation is given by right multiplication

$$A'^+ = (\widehat{hs})^{-1} d(\widehat{hs}), \quad A'^- = (\widehat{gr})^{-1} d(\widehat{gr}) \tag{3.118}$$

One can calculate the components of (3.116) in detail using the relations (3.110), e.g. for the A^+ part we get

$$\begin{aligned} A^+ &= \left(h_4 \mathbb{1}_4 - h_i J_i^+\right)\left(dh_4 \mathbb{1}_4 + dh_k J_k^+\right) \\ &= h_4 dh_4 \mathbb{1}_4 + \left(h_4 dh_i - h_i dh_4\right) J_i^+ - h_i dh_k J_i^+ J_k^+ \\ &= \left(\delta_{ab} h_a dh_b\right) \mathbb{1}_4 + \left(h_4 dh_i - h_i dh_4\right) J_i^+ - \epsilon_{ikl} h_i dh_k J_l^+ \\ &= \left(h_4 dh_i - h_i dh_4 - \epsilon_{ikl} h_k dh_l\right) J_i^+ \end{aligned} \tag{3.119}$$

In the last step we used $\delta_{ab} h_a h_b = 1$. The calculation of the A^- part is completely analogous. Finally, the triad reads

$$e_i = h_4 dh_i - g_4 dg_i + g_i dg_4 - h_i dh_4 + \epsilon_{ikl}(g_k dg_l - h_k dh_l) \tag{3.120}$$

From this, the metric can be calculated as $ds^2 = e_i e_i$, but the formula does not look very inspiring. However, if one of the factors is trivial, e.g. $g = \mathbb{1}_2$, the metric takes the simple form

$$ds^2 = dh_1^2 + dh_2^2 + dh_3^2 + dh_4^2 \tag{3.121}$$

If h is the identity map,

$$\begin{aligned} h : S^3 &\to SU(2) \cong S^3 \\ X &\mapsto X \end{aligned} \tag{3.122}$$

(not to be confused with $h = \mathbb{1}_2$, which we call a trivial gauge transformation), and $X = (x_1, x_2, x_3, x_4)$ are coordinates from the embedding of S^3 in $\mathbb{R}^4$, then the metric is just the restriction to S^3 of the Euclidean metric on $\mathbb{R}^4$

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \tag{3.123}$$

Gauge transforming the vacuum

It is clear that starting from the field configuration $A = 0$ (which we call the gauge theory vacuum), one can produce any gauge field which solves $F_A = 0$ by a gauge transformation. In most cases, the vielbein component will be degenerate. If it isn't, the vielbein and the spin connection together define a geometry which locally looks like S^3, but may have a different global structure. Moreover, this geometry is invariant under the subgroup H of local frame rotations.

Let us explore this terrain a bit. One class of gauge transformations is given by

$$\begin{aligned} g &= \cos(n\psi)\mathbb{1}_2 + i\sin(n\psi)\left[\cos(\theta)\sigma_1 + \sin(\theta)\left(\cos(\phi)\sigma_2 + \sin(\phi)\sigma_3\right)\right] \\ h &= \cos(m\psi)\mathbb{1}_2 + i\sin(m\psi)\left[\cos(\theta)\sigma_1 + \sin(\theta)\left(\cos(\phi)\sigma_2 + \sin(\phi)\sigma_3\right)\right] \end{aligned} \tag{3.124}$$

where ψ, θ, ϕ are hyperspherical coordinates on S^3, having the ranges

$$0 \leq \psi, \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi \tag{3.125}$$

and $m, n \in \mathbb{Z}$. The maps g and h wrap around SU(2) a certain number of times. Indeed, the $SU(2)$ winding numbers turn out to be

$$\begin{aligned} W(g) &= \frac{1}{24\pi^2}\int_{S^3} tr\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right) = n \\ W(h) &= \frac{1}{24\pi^2}\int_{S^3} tr\left(h^{-1}dh \wedge h^{-1}dh \wedge h^{-1}dh\right) = m \end{aligned} \tag{3.126}$$

If we cook up the $SO(4)$ gauge transformation $R(g,h)$, it can also be assigned a winding number

$$W(R) = \frac{-1}{48\pi^2}\int_{S^3} tr\left(R^{-1}dR \wedge R^{-1}dR \wedge R^{-1}dR\right) = n + m \tag{3.127}$$

and the result $n+m$ follows from the fact that $R(g,h)$ splits according to equation (3.115). The different normalizations in front of the winding number integrals are due to the relative factors of the trace operators in $\mathfrak{su}(2)$ and $\mathfrak{so}(4)$ respectively.

Let us proceed to the calculation of the triad. The components g_i and h_i can be read off directly from (3.124), and the vielbein can be calculated from equation (3.120),

$$\begin{aligned} e_1 &= (m-n)\cos\theta d\psi + \tfrac{1}{2}\sin\theta\left(\sin(2n\psi) - \sin(2m\psi)\right)d\theta \\ &\quad - \tfrac{1}{2}\sin^2\theta\left(\cos(2n\psi) - \cos(2m\psi)\right)d\phi \\ e_2 &= \ldots \\ e_3 &= \ldots \end{aligned} \tag{3.128}$$

which results in very long and uninteresting expressions. On the other hand, the metric looks very nice

$$ds^2 = (m-n)^2 d\psi^2 + \sin^2(|m-n|\,\psi)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{3.129}$$

and for $m = 1$ and $n = 0$ (or vice versa) reproduces the 3-sphere metric in spherical coordinates. For $m = n$, this metric is zero and thus completely degenerate while the winding number of the gauge transformation is $2n$. If $m = -n$, the winding number is zero, but the metric is non-degenerate if $n \neq 0$.

By applying gauge transformations, one can change between these different geometries. On the other hand, two gauge transformations having distinct winding numbers are in a different homotopy class and cannot be continuously deformed into each other. Therefore they can be thought to represent different configurations. Please have a look at [5] for further considerations on the nature of gauge transformations.

Rebuilding and transforming the Hopf metric

Another coordinate patch which covers all of S^3 (except for the poles) is given by the Euler angles

$$0 \le \psi \le 2\pi, \quad 0 \le \theta \le \pi/2, \quad 0 \le \phi \le \pi \tag{3.130}$$

and we can define a gauge field by picking

$$h = e^{i\phi\sigma_3} e^{i\theta\sigma_2} e^{i\psi\sigma_3}, \quad g = \mathbb{1}_2 \tag{3.131}$$

This is of course just the identity map expressed in Euler coordinates. The coefficients of the representation $h = h_4\mathbb{1}_2 + ih_i\sigma_i$ are

$$\begin{aligned} h_1 &= \sin(\theta)\sin(\phi-\psi) \\ h_2 &= \sin(\theta)\cos(\phi-\psi) \\ h_3 &= \cos(\theta)\sin(\phi+\psi) \\ h_4 &= \sin(\theta)\cos(\phi+\psi) \end{aligned} \tag{3.132}$$

and from these and (3.120), one can directly compute the triad

$$\begin{aligned} e_1 &= -\sin(2\theta)\cos(2\phi)d\psi + \sin(2\phi)d\theta \\ e_2 &= \sin(2\theta)\sin(2\phi)d\psi + \cos(2\phi)d\theta \\ e_3 &= \cos(\theta)d\psi + d\phi \end{aligned} \tag{3.133}$$

and the metric

$$ds^2 = d\psi^2 + d\theta^2 + d\phi^2 + 2\cos(2\theta)d\psi d\phi \tag{3.134}$$

This constitutes a perfectly sound geometry[8], apart from the usual coordinate singularities. The determinant of the dreibein is $\det e = -\sin(2\theta)$, so it is invertible on the whole sphere except for the poles ($\theta = 0$ or $\theta = \pi/2$).

In this setting, it is very obvious how to make the geometry degenerate. Since according to (3.117) gauge transformations are composed by right multiplication, we can immediately list some transformations which remove one of the coordinates from the metric and thus make it degenerate in one or more dimensions. Let us list some examples

$$\begin{aligned} s = e^{-i\psi\sigma_3} \quad &\rightsquigarrow \quad ds^2 = d\theta^2 + d\phi^2 \\ s = e^{-i\psi\sigma_3}e^{-i\theta\sigma_2}e^{i\psi\sigma_3} \quad &\rightsquigarrow \quad ds^2 = (d\psi + d\phi)^2 \\ s = h^{-1}e^{i\theta\sigma_2}e^{i\psi\sigma_3} \quad &\rightsquigarrow \quad ds^2 = d\psi^2 + d\theta^2 \end{aligned} \tag{3.136}$$

[8] In fact, the metric can also be written as

$$ds^2 = (d\theta^2 + \sin^2\theta d\phi^2) + (d\psi + B)^2 \tag{3.135}$$

where $B = \cos\theta d\phi$ is a $U(1)$ - connection. This way of writing the 3-sphere metric exhibits the fact that S^3 is a principal S^1-bundle over S^2, called the Hopf fibration. The metric is therefore referred to as the Hopf metric, whose first term is the metric on the base space S^2 while the presence of the 'gauge field' B in the second term is due to the non-trivial structure group.

4 MacDowell-Mansouri gravity

We proceed in the same spirit as in dimension three to set up a gauge theory of gravity also in four spacetime dimensions, which is a more realistic model of the universe we live in. In fact, gravity in dimension four is much more complicated. Therefore the homogeneous models are only an approximation to our spacetime. The connection describing the geometry is no longer necessarily flat.

The biggest problem however is finding an action which serves the same purpose as the Chern-Simons action did in three dimensions, namely reduce to the Palatini action in a suitable parametrization. In this chapter, one attempt at this will be presented, namely the MacDowell-Mansouri approach.

4.1 Kinematics

Again, we want to study spacetimes with positive or negative cosmological constant and of Riemannian and Lorentzian signature. Thus, there are again six distinct classes of models, summarized in table 4.1, and each of them will have a Cartan geometry based on one of the six homogeneous spaces. So most of the kinematical setting is built in exactly the same way as in three dimensions.

The ambient metric is chosen to be

$$\eta_{IJ} = diag(\eta', 1, 1, 1, k) \tag{4.1}$$

where k is the sign of the cosmological constant Λ and η' is the determinant of the upper 4×4 metric η_{ab}. Upper-case indices $I, J, ..$ range from 0 to 4 and $a, b, ...$ from 0 to 3 (we do not distinguish between Riemannian and Lorentzian indices for convenience). So the Lie subalgebra $\mathfrak{h}$ is $\mathfrak{so}(4)$ for the Riemannian spaces and $\mathfrak{so}(3,1)$ for the Lorentzian ones. The gauge field is split according to

$$A = \omega + \tfrac{1}{l} e \tag{4.2}$$

Since the Lie algebra is now 10-dimensional, it is actually easier to work in components instead of generators and commutation relations. The splitting

	$\Lambda < 0$	$\Lambda = 0$	$\Lambda > 0$
Riemannian	EAdS $SO(4,1) / SO(4)$	Euclidean $ISO(4) / SO(4)$	EAdS $SO(5) / SO(4)$
Lorentzian	AdS $SO(3,2) / SO(3,1)$	Minkowski $ISO(3,1) / SO(3,1)$	dS $SO(4,1) / SO(3,1)$

Table 4.1: Homogeneous model spacetimes in dimension four

of the gauge field reads

$$A^I{}_J \left\langle \begin{array}{l} A^a{}_b = \omega^a{}_b \\ \\ A^a{}_4 = \frac{1}{l} e^a \end{array} \right. \tag{4.3}$$

and of course all diagonal components vanish. The spin connection ends up in the upper left block of A, which corresponds to $\mathfrak{h}$ as it should. Moreover,

$$A^4{}_a = \eta^{44} A_{4a} = -\eta^{44} A_{a4} = -kl^{-1} e_a \tag{4.4}$$

and from this, the components of the field strength can be calculated explicitly. The $\mathfrak{h}$-valued part is once again given by the corrected curvature

$$\begin{aligned} \hat{F}^a_b &= dA^a{}_b + A^a{}_c \wedge A^c{}_b + A^a{}_4 \wedge A^4{}_b \\ &= d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b - kl^{-2} e^a \wedge e_b \\ &= \Omega^a{}_b - kl^{-2} e^a \wedge e_b \end{aligned} \tag{4.5}$$

and the other piece by the torsion

$$\begin{aligned} F^a_4 &= dA^a{}_4 + A^a{}_c \wedge A^c{}_4 \\ &= \tfrac{1}{l} \left(de^a + \omega^a{}_c \wedge e^c \right) \\ &= \tfrac{1}{l} T^a \end{aligned} \tag{4.6}$$

and $F^4_a = -kl^{-1} T_a$, such that we can write

$$F = \hat{F} + \tfrac{1}{l} T, \quad \hat{F} = \Omega - kl^{-2} e^2 \tag{4.7}$$

as in the three dimensional case. Using the convention for gauge covariant derivatives introduced in equation (3.9), we introduce D_ω for the $\mathfrak{h}$-valued parts of tensors and connections. The Bianchi identity for F then reduces to

$$D_A F = 0 \quad \Rightarrow \left\langle \begin{array}{l} D_\omega \Omega = 0 \\ \\ D_\omega T = D^2_\omega e = \Omega \wedge e \end{array} \right. \tag{4.8}$$

4.2 The MacDowell-Mansouri action

The approach of MacDowell and Mansouri [3] to write down an action for four dimensional gravity was

$$S_{\mathrm{MM}}[A] = \int \epsilon_{abcd} \hat{F}^{ab} \wedge \hat{F}^{cd} \tag{4.9}$$

The first remark on this approach is that the action is not gauge invariant! While it is invariant under the subgroup $\mathcal{H}$, the full gauge group mixes the hatted field strength with the torsion. It is however possible to set up a gauge invariant theory which reduces to the MacDowell-Mansouri action via a constraint, and this will be the topic of the next section. For now, let us work in this 'broken phase' and see what happens.

When the action is expressed in terms of the constituent fields

$$\begin{aligned} S_{\mathrm{MM}} &= \int \epsilon_{abcd} \Omega^{ab} \wedge \Omega^{cd} + l^{-4} \int \epsilon_{abcd} e^{abcd} \\ &\quad - 2kl^{-2} \int \epsilon_{abcd} \Omega^{ab} \wedge e^{cd} \\ &= -4kl^{-2} S_{\mathrm{Pal}} + \int \epsilon_{abcd} \Omega^{ab} \wedge \Omega^{cd} \end{aligned} \tag{4.10}$$

where

$$S_{\mathrm{Pal}} = \tfrac{1}{2} \int \epsilon_{abcd} \left(\Omega^{ab} \wedge e^{cd} - \tfrac{1}{2} k l^{-2} e^{abcd} \right) \tag{4.11}$$

The Ω^2 term is topological because of the Bianchi identity $D_\omega \Omega = 0$

$$\begin{aligned} \delta \int \epsilon_{abcd} \Omega^{ab} \wedge \Omega^{cd} &= 2 \int \epsilon_{abcd} \Omega^{ab} \wedge \delta \Omega^{cd} \\ &= 2 \int \epsilon_{abcd} \Omega^{ab} \wedge D_\omega \delta \omega^{cd} \\ &= -2 \int \epsilon_{abcd} \cancel{D_\omega \Omega^{ab}} \wedge \delta \omega^{cd} = 0 \end{aligned} \tag{4.12}$$

Thus, the MM action is equivalent to the Palatini action for the cases with nonzero cosmological constant and is purely topological for $k = 0$.

4.3 The Stelle-West model

The MacDowell-Mansouri action is only invariant under the group $\mathcal{H}$, and the aim is to write down an action which is invariant under the full group $G = SO(5)$, $SO(4,1)$ or $SO(3,2)$ and have this symmetry spontaneously broken by a field ϕ in the fundamental representation

$$\phi^I \to g^I{}_J \phi^J \tag{4.13}$$

which satisfies the constraint $\eta_{IJ}\phi^I\phi^J = k$. This is achieved by choosing the action

$$S_{\mathrm{SW}} = \int \epsilon_{IKLMN} F^{IK} \wedge F^{LM} \wedge \phi^N + S_C^{(1)} \tag{4.14}$$

where

$$S_C^{(1)} = \frac{k}{2} \int \sigma \left(1 - k\eta_{IJ}\phi^I\phi^J\right) \tag{4.15}$$

is a constraint action, with σ a Lagrange multiplier field (a volume form). Variation with respect to this form imposes the constraint

$$\frac{\delta S_{\mathrm{SW}}}{\delta\sigma} = 0 \quad \Rightarrow \quad \eta_{IJ}\phi^I\phi^J = k \tag{4.16}$$

whereas the variation with respect to the scalar field ϕ yields

$$\frac{\delta S_{\mathrm{SW}}}{\delta\phi^I} = 0 \quad \Rightarrow \quad \epsilon_{IKLMN} F^{KL} \wedge F^{MN} = \sigma\eta_{IJ}\phi^J \tag{4.17}$$

After multiplication by ϕ^I and rearrangement, this equation takes the form

$$\sigma = k\epsilon_{IKLMN} F^{KL} \wedge F^{MN} \wedge \phi^I \tag{4.18}$$

which means that the Lagrange multiplier field is fixed dynamically[1]. In a gauge in which $\phi^I = \delta^I{}_4$, we can choose $\epsilon_{abcd4} = \epsilon_{abcd}$ and the action reduces to the MacDowell-Mansouri action

$$S_{\mathrm{SW}} \quad \rightarrow \quad \int \epsilon_{abcd} F^{ab} \wedge F^{cd} = S_{\mathrm{MM}} \tag{4.19}$$

Moreover, it follows from the Einstein equations that σ is proportional to the Weyl tensor squared. In fact, towards the end of section 2.5 we derived[2]

$$\Omega^c = 3kl^{-2}e^c, \quad \mathcal{R} = 12kl^{-2} \tag{4.20}$$

If this is substituted into the Weyl 2-form, given (in dimension four) by

$$W^{ab} = \Omega^{ab} - \tfrac{1}{2}\left(e^a \wedge \Omega^b - e^b \wedge \Omega^a\right) + \tfrac{1}{6}\mathcal{R}e^a \wedge e^b \tag{4.21}$$

it turns out that

$$\begin{aligned} W^{ab} &= \Omega^{ab} - 3kl^{-2}e^{ab} + 2kl^{-2}e^{ab} \\ &= \Omega^{ab} - kl^{-2}e^{ab} \\ &= \hat{F}^{ab} \end{aligned} \tag{4.22}$$

[1] The field ϕ is not really dynamical, nor is σ, since the action lacks any derivative terms involving those fields. Thus the construction remains somewhat artificial.

[2] This derivation only works for invertible vielbeins, because it is only in this case that the dual operators i_a exist. But the Weyl tensor is also constructed using contractions, so in any case, the correspondence which we are after makes sense only if the vielbein is non-singular.

Thus, on-shell and if the vielbein is invertible, the corrected curvature equals the Weyl 2-form, and therefore the Lagrange multiplier is given by

$$\sigma = k\epsilon_{abcd} W^{ab} \wedge W^{cd} \tag{4.23}$$

What is this fuss about the Weyl tensor? The important thing is that the field equation from $\delta\phi^I$ does not contradict Einstein's equations. In fact, this happens if instead of $S_C^{(1)}$ one uses

$$S_C^{(m)} = \frac{k}{2} \int \sigma \left(1 - k\eta_{IJ}\phi^I\phi^J\right)^n, \quad m \geq 2 \tag{4.24}$$

as a constraint action. The variation of σ gives rise to the same constraint as before, but the ϕ field equation now reads

$$m\sigma \left(1 - k\eta_{IJ}\phi^I\phi^J\right)^{m-1} = k\epsilon_{IKLMN} F^{KL} \wedge F^{MN} \wedge \phi^I \tag{4.25}$$

Because of the constraint, the left-hand side vanishes on-shell (if $m \geq 2$). In the gauge we chose above and if the vielbein is invertible, we therefore get an additional constraint on the Weyl tensor, namely

$$\epsilon_{abcd} W^{ab} \wedge W^{cd} = 0 \tag{4.26}$$

which is not required by Einstein's equations.

A Appendices

A.1 Lie algebra conventions for Chern-Simons gravity

We give the conventions for the Lie algebras $\mathfrak{g} = \mathfrak{so}(4)$, $\mathfrak{so}(2,2)$ and $\mathfrak{so}(3,1)$ in detail. The generators J_{ab} are chosen to be

$$(J_{ab})^c{}_d = \eta_{ad}\delta^c_b - \eta_{bd}\delta^c_a \tag{A.1}$$

such that they satisfy commutation relations

$$[J_{ab}, J_{cd}] = \eta_{ac}J_{bd} - \eta_{bc}J_{ad} - \eta_{ad}J_{bc} + \eta_{bd}J_{ac} \tag{A.2}$$

where η_{ab} is the metric tensor preserved by the group. Please note that indices $a, b, c, ...$ range from 1 to 4 and $J_{ab} = -J_{ba}$, so we have six independent generators. We can further define a Hodge star operator which acts as

$$A^a{}_b \longmapsto \star A^a{}_b = \tfrac{1}{2}\epsilon^a{}_{bc}{}^d A^c{}_d \tag{A.3}$$

Let's act on one of the basis vectors J_{ab}. We get

$$\begin{aligned} \star\,(J_{ab})_{cd} &= \tfrac{1}{2}\epsilon_{cdef}\,(J_{ab})^{ef} = \tfrac{1}{2}\epsilon_{abef}\left(J^{ef}\right)_{cd} \\ \star J_{ab} &= \tfrac{1}{2}\epsilon_{abcd}J^{cd} \end{aligned} \tag{A.4}$$

Moreover, the star operator squares to the determinant of the metric

$$\star^2 = \eta \tag{A.5}$$

and it commutes with the adjoint action of the Lie algebra, as we can calculate from the commutation relations[1]

$$\begin{aligned} \star\,[J_{ab}, \star J_{cd}] &= \star\tfrac{1}{2}\epsilon_{cdef}\left[J_{ab}, J^{ef}\right] \\ &= \star\tfrac{1}{2}\epsilon_{cdef}(\eta^e_a J_b{}^f - ...) \\ &= \tfrac{1}{4}\epsilon_{cdef}\epsilon_{bgh}{}^f\eta^e_a J^{gh} - ... \\ &= \eta(\eta_{cb}\eta_{dg}\eta_{eh} + ...)\eta^e_a J^{gh} \\ &= -\eta\eta_{bc}J_{ad} + ... \\ &= \star^2\,[J_{ab}, J_{cd}] \end{aligned} \tag{A.6}$$

[1]The ellipses indicate that there are other combinations of indices determined by the symmetries of the left-hand side of the equation.

and hence

$$\star [X, Y] = [X, \star Y] = [\star X, Y] , \quad X, Y \in \mathfrak{g} \tag{A.7}$$

In terms of the vector basis

$$J_i = \tfrac{1}{2}\epsilon_{ijk} J^{jk}, \quad P_i = \star J_i \tag{A.8}$$

the commutation relations can be easily calculated using the Hodge operator. One only needs to compute

$$\begin{aligned}
[J_i, J_j] &= \tfrac{1}{4}\epsilon_{ikl}\epsilon_{jmn}\left[J^{kl}, J^{mn}\right] \\
&= \tfrac{1}{4}\epsilon_{ikl}\epsilon_{jmn}(\eta^{km}J^{ln} - ...) \\
&= \epsilon_{kli}\epsilon^{k}{}_{nj}J^{ln} \\
&= \eta'(\eta_{ln}\eta_{ij} - \eta_{lj}\eta_{in})J^{ln} \\
&= \eta' J_{ij} \\
&= \epsilon_{ijk}J^{k}
\end{aligned} \tag{A.9}$$

and the other commutators are given by

$$\begin{aligned}
[J_i, P_j] &= [J_i, \star J_j] \\
&= \star [J_i, J_j] \\
&= \epsilon_{ijk}P^{k}
\end{aligned} \tag{A.10}$$

and

$$\begin{aligned}
[P_i, P_j] &= [\star J_i, \star J_j] \\
&= \star [\star J_i, J_j] \\
&= \star^2 [J_i, J_j] \\
&= \eta\epsilon_{ijk}J^{k}
\end{aligned} \tag{A.11}$$

A.2 SO(4) covering map

The vector space $\mathbb{R}^4$ is isomorphic to the space of quaternions, which can be represented by 2×2 complex matrices

$$\begin{array}{ccc}
\mathbb{R}^4 & \rightarrow & \mathbb{H} \\
x_a & \mapsto & x_4 \mathbb{1}_2 + X_i T_i = X
\end{array} \tag{A.12}$$

where $T_i = i\sigma_j$. The modulus of the four-vector in $\mathbb{R}^4$ is given by the determinant of the quaternionic matrix

$$\delta_{ab}x_a x_b = \det X \tag{A.13}$$

The length of a four-vector is invariant under $SO(4)$. Similarly, the determinant is unchanged under $X \mapsto h^\dagger X g$, with $g, h \in SU(2)$. The converse is also true, if $X, X' \in \mathbb{H}$ with $\det X = \det X'$, then those two matrices can be diagonalized by $SU(2)$ matrices,

$$U_{11}^\dagger X U_{12} = U_{21}^\dagger X' U_{22} = D \tag{A.14}$$

and therefore there are $g, h \in SU(2)$ such that $X' = h^\dagger X g$. They are not unique however, since $-g, -h$ constitute another solution. This is of no particular importance, since we want to find a transformation $R(g, h)$ such that the diagram

$$\begin{array}{ccc} \mathbb{H} & \longrightarrow & \mathbb{R}^4 \\ {\scriptstyle h^\dagger()g}\big\downarrow & & \big\downarrow{\scriptstyle R(g,h)} \\ \mathbb{H} & \longrightarrow & \mathbb{R}^4 \end{array} \tag{A.15}$$

commutes. Using the representation

$$g = g_4 \mathbb{1}_2 + g_i T_i, \quad h = h_4 \mathbb{1}_2 + h_i T_i \tag{A.16}$$

and the relations

$$\begin{aligned} T_i T_k &= -\delta_{ik} \mathbb{1}_4 + \epsilon_{ikl} T_l \\ T_i T_k T_l &= -\epsilon_{ikl} \mathbb{1}_4 - \delta_{ik} T_l - \delta_{kl} T_i + \delta_{il} T_k \end{aligned} \tag{A.17}$$

one can calculate

$$X' = h^\dagger X g = x'_4 \mathbb{1}_2 + x'_i T_i \tag{A.18}$$

and read off the coefficients of the $SO(4)$ matrix $R(g, h)$, defined by

$$x'_a = R_{ab} x_b \tag{A.19}$$

The result is

$$\begin{aligned} R_{44} &= h_4 g_4 + h_i g_i \\ R_{i4} &= h_4 g_i - g_4 h_i + \epsilon_{ikl} h_k g_l \\ R_{4i} &= -h_4 g_i + g_4 h_i + \epsilon_{ikl} h_k g_l \\ R_{ij} &= (h_4 g_4 - h_k g_k)\delta_{ij} + h_i g_j + h_j g_i - \epsilon_{ijk}(h_4 g_k + h_k g_4) \end{aligned} \tag{A.20}$$

If $R(g, h)$ is supposed to be a product of matrices $\hat{h}\tilde{g}$ where each factor only depends on its corresponding $SU(2)$ counterpart, the components of $\hat{h}, \tilde{g}$ can be guessed from the components of R. Indeed, the first line of (A.20) determines (up to signs) the last coloumn of $\tilde{g}$ and the last row of $\hat{h}$,

$$R = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ h_1 & h_2 & h_3 & h_4 \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & g_1 \\ \cdot & \cdot & \cdot & g_2 \\ \cdot & \cdot & \cdot & g_3 \\ \cdot & \cdot & \cdot & g_4 \end{pmatrix} \tag{A.21}$$

From here on we are basically solving some kind of Sudoku. From the second and third lines in (A.20) we can deduce all entries

$$R = \begin{pmatrix} h_4 & -h_3 & h_2 & -h_1 \\ h_3 & h_4 & -h_1 & -h_2 \\ -h_2 & h_1 & h_4 & -h_3 \\ h_1 & h_2 & h_3 & h_4 \end{pmatrix} \begin{pmatrix} g_4 & -g_3 & g_2 & g_1 \\ g_3 & g_4 & -g_1 & g_2 \\ -g_2 & g_1 & g_4 & g_3 \\ -g_1 & -g_2 & -g_3 & g_4 \end{pmatrix} \tag{A.22}$$

and the last equation then is automatically solved, whereas any other sign choice in (A.21) would have been inconsistent.

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